1 Wiener’s Filter

\[ e[n] = D[n] - Y[n] \]

Want to minimize \( E[e^2[n]] \) by selecting \( W_i \)’s where,

\[ Y[n] = \sum_{i=0}^{k} W_i X[n - i] \]

\[ W = R^{-1}_{XX} P \]

where,

\[ P = R_{DX} \triangleq \begin{bmatrix} E[D[n]X[n]] \\ E[D[n]X[n-1]] \\ \vdots \\ E[D[n]X[n-k]] \end{bmatrix} \]

The optimal (smallest) \( E[e^2[n]] \) is:

\[ E[e^2[n]] = E[(D[n] - Y[n])^2] \]

\[ = E[D^2[n]] - 2E[D[n]Y[n]] + E[Y^2[n]] \]

\[ = E[D^2[n]] - 2E[(e[n] + Y[n])Y[n]] + E[Y^2[n]] \]

\[ = E[D^2[n]] - E[Y^2[n]] - 2E[e[n]Y[n]] \]

"Every problem is an optimization problem in disguise."

–Anonymous
\[ E[D^2[n]] - E[Y^2[n]] - 2E[e[n]] \sum_{i=0}^{k} W_i X[n-i] \]

\[ = E[D^2[n]] - E[Y^2[n]] - 2 \sum_{i=0}^{k} W_i E[e[n]X[n-i]] \]

\[ = E[D^2[n]] - E[Y^2[n]] \]

\[ = \sigma_D^2 - \sigma_Y^2 \]

Note: \( R_{YY}[m] = E[Y[n + m]Y[n]] \Rightarrow E[Y^2[n]] = R_{YY}[0] \)

\[ S_{YY}(\omega) = \sum_{m=-\infty}^{\infty} R_{YY}[m]e^{-j\omega m} = \sum_{m=-\infty}^{\infty} R_{YY}[m] \]

Example

\[ R_{XX} = \begin{bmatrix} R_{XX}[0] & R_{XX}[1] \\ R_{XX}[1] & R_{XX}[0] \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \]

\[ P = \begin{bmatrix} 0.5 \\ 0.75 \end{bmatrix} \triangleq \begin{bmatrix} R_{DX}[0] \\ R_{DX}[1] \\ \vdots \\ R_{DX}[k] \end{bmatrix} \]

Thus,

\[ W = R_{XX}^{-1}P = \frac{1}{1 - 0.25} \begin{bmatrix} 0.5 \\ 0.75 \end{bmatrix} \]

Example  
Prediction: Suppose \( X[n] = W[n] + 0.5W[n-1] \) where \( W[n] \) is i.i.d with \( \sigma_w^2 = 1 \). Predict \( X[n] \) based on \( X[n-1] \) and \( X[n-2] \).
\[ R_{XX}[m] = E[X[n+m]X[n]] = E[(W[n+m]+0.5W[n+m-1])(W[n]+0.5W[n-1])] \]
\[ = R_{WW}[m] + 0.5R_{WW}[m+1] + 0.5R_{WW}[m-1] + 0.25R_{WW}[m] \]
\[ = 1.25\sigma[m] + 0.5\sigma[m+1] + 0.5\sigma[m-1] \]

\[ R_{DX}[m] = ? \]

\[ D[n] = X[n] \]
\[ P = R_{DX} = \begin{bmatrix} E[X[n]X[n-1]] \\ E[X[n]X[n-2]] \end{bmatrix} = \begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \end{bmatrix} \]
\[ R_{XX} = \begin{bmatrix} R_{XX}[0] & R_{XX}[1] \\ R_{XX}[1] & R_{XX}[0] \end{bmatrix} = \begin{bmatrix} 1.25 & 0.5 \\ 0.5 & 1.25 \end{bmatrix} \]
\[ R_{DX} = P = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \]

Thus,

\[ W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 1.25 & 0.5 \\ 0.5 & 1.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.48 \\ -0.19 \end{bmatrix} \]

\[ X[n] = W_1 X[n-1] + W_2 X[n-2] \]

\section{Synthesis of Random Sequence}

Figure 1: Synthesis of Random Sequence

Figure 1 shows the shape of a synthesized Random Sequence.

White noise:  \begin{align*} 
& \text{(1) } i.i.d \\
& \text{(2) } E[W[n]] = 0 
\end{align*} \Rightarrow S_{WW}(w) = C
\[ R_{WW}[m] = \sigma_W^2 \delta[m] \]
Our goal is to figure out $h[n]$ so that $S_{XX}(w)$ matches a given spectral density.

$$S_{XX}(z) = S_{WW}(z)H(z)H^*(\frac{1}{z})$$

If $h[n]$ is real then

$$S_x(z) = H(z)H(\frac{1}{z})S_{WW}(z) = H(z)H(\frac{1}{z})\sigma_W^2$$

The idea is to factor $S_{XX}(z)$ into $H(z)$ and $H(\frac{1}{z})$, so that we can obtain $H(z)$.

Next, invert $H(z)$ to $h[n]$, $h[n]$ is your desired filter with the input is white noise.

**Example**

$$S_{XX}(\omega) = \frac{5 + 4 \cos 2\omega}{10 + 6 \cos \omega} = \frac{5 + 4(e^{j2\omega} + e^{-j2\omega})}{10 + 6(e^{j2\omega} + e^{-j2\omega})} = \frac{5 + 2e^{j2\omega} + 2e^{-j2\omega}}{10 + 3e^{j2\omega} + 3e^{-j2\omega}} = \frac{5 + 2z^2 + 2z^{-2}}{10 + 3z + 3z^{-1}}$$

The reason I choose $H(z) = \left[\frac{2(1 + \frac{1}{3}z^{-2})}{3(1 + \frac{1}{3}z^{-1})}\right]$, because $H(z)$ will be causal and stable.

This is because the pole is at $-\frac{1}{3}$ which is inside the unit circle $\Rightarrow$ stable.

$$h[n] = \frac{2}{3}(-\frac{1}{3})^nu[n] + \frac{1}{3}(-\frac{1}{3})^{n-2}u[n-2]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

### 3 Channel Equalization

![Figure 2: The Power Spectral of Original Input Signal $S_{XX}(w)$](image-url)
After the original signal in figure 2 goes through the channel in figure 3, the passband signal has few power, shown in figure 4.

Figure 3: The Channel of Low Pass Filter.

Figure 4: The Power Spectral of Output Signal $S_Y(w)$.

After the original signal in figure 2 goes through the channel in figure 3, the passband signal has few power, shown in figure 4.

Figure 5: The Original Signal $S_{XX}(w)$ After Whitening.
Then, with the whitening transformation of the original signal, we get the uncorrelated signal $X(n)$, which power spectral is constant as shown in figure 5. Meanwhile, the new signal goes through the same channel. The output in figure 6 includes more power. The reason why we do the channel equalization is that when the signal has more power, the SNR is higher. This results most your signal at around DC gets wiped out.

![Figure 6: The New Power Spectral of Output Signal $S_Y(w)$.](image)

In the figure 7, the LTI system a whitening filter, which transfers correlated $X[n]$ to uncorrelated $W[n]$.

![Figure 7: Whitening Filter](image)

In the figure 7, the LTI system a whitening filter, which transfers correlated $X[n]$ to uncorrelated $W[n]$.

![Figure 8: The Signal $W[n]$ Recovering](image)
Usually, we know the recovering signal $W[n]$ and the power spectral $S_Y(w)$, but we don’t know what the channel exactly is. The figure 8 shows a method to solve the problem. With mathematical computing, we could find the transfer function of $H(z)$. Also, there is a whitening filter at receiver $1/H(z)$.

**Example** $S_{YY}(w) = 10 + 6\cos(w)$, Find whitening filter that make the output looks like white noise.

$$S_{YY}(w) = 10 + 6\cos(w) = 10 + 6[(e^{jw} + e^{-jw})/2]$$

$$S_{YY}(z) = 10 + 3(z + z^{-1})$$

Note the following:

$$\sigma^2(1 + az)(1 + az^{-1}) = \sigma^2(1 + a^2) + \sigma^2a(z + z^{-1})$$

$$10 = \sigma^2(1 + a^2) \quad \text{and} \quad 3 = \sigma^2a$$

$$\Rightarrow \frac{1+a^2}{a} = \frac{10}{3} \Rightarrow 3a^2 - 10a + 3 = 0 \Rightarrow \begin{cases} a = \frac{1}{3}, \sigma = 3 \\ a = 3, \sigma = 1 \end{cases}$$

$$S_{XX}(z) = 9(1 + \frac{1}{3}z^{-1})(1 + \frac{1}{3}z)$$

$$H(z) = (1 + z^{-1}) \text{ for stable and causal.}$$

So whitening filter,

$$\frac{1}{H(z)} = \frac{1}{1 + \frac{1}{3}z^{-1}} \Rightarrow h[n] = (-\frac{1}{3})^n u[n]$$