\[ y[n] = 3y[n-1] + y[n-2] + x[n-1] \]

Applying z-transform on both sides of the equation we get

\[ Y(z) = 3z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z) \]

\[ \Rightarrow Y(z) \left[ 1 - 3z^{-1} - z^{-2} \right] = z^{-1}X(z) \]

\[ \Rightarrow \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 3z^{-1} - z^{-2}} = \frac{z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right) - 2z^{-1} - \left(1 - \frac{1}{2}z^{-1}\right)} \]

\[ = \frac{z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} = H(z) \]

Poles at \( z = -\frac{1}{2}, 2 \)
Zero at \( z = 0, 0 \)

Because the system is causal the ROC would be \( |z| > 2 \).

As the ROC of the system function does not include the unit circle, system is unstable.

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} = \frac{\left(1 + \frac{1}{2}z^{-1}\right) - 1 + 2z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} \times \frac{2}{5} \]

\[ = \frac{2}{5} \left( \frac{1}{1 - 2z^{-1}} \right) - \frac{2}{5} \left( \frac{1}{1 + \frac{1}{2}z^{-1}} \right) \]

The inverse z-transform will give us the impulse response of the system

\[ h[n] = z^{-1}\{H(z)\} = \frac{2}{5}(2^n)u[n] - \frac{2}{5} \left(\frac{1}{2}\right)^n u[n] \]
c) the achieved system function that is consistent with the difference equation can draw three different ROC.

**Case-1**

System would be causal but not stable.

**Case-2**

In this case the system would be noncausal and stable.

**Case-3**

The system would be noncausal and unstable.
So the stable (zero causal) impulse response that satisfies the difference equation will have the ROC similar to case 2.

The corresponding impulse response would be:

$$h[n] = (-\frac{1}{2})^n(-\frac{1}{2}u[n] \rightarrow (2/3)(2)^n u[-n-1]$$
\( y[n] = (\frac{1}{3})^n u[n] + (\frac{1}{4})^n u[n] + u[n] \)

Applying the Z-transform on both sides of the equation we get:

\[
Y(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{1}{1 - z^{-1}}
\]

\[
= \frac{(-\frac{1}{4}z^{-1})(1-z^{-1}) + (1 - \frac{1}{3}z^{-1})(1-z^{-1}) + (1 - \frac{1}{4}z^{-1})(1-z^{-1})}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})(1-z^{-1})}
\]

\[
= \frac{[1-z^{-1} - \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2}] + [1-z^{-1} - \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2}] + [1-z^{-1} - \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2}] - \frac{1}{12}z^{-2}}{1 - 2z^{-1} - \frac{3}{12}z^{-2} + \frac{7}{12}z^{-3} + \frac{1}{12}z^{-2} - \frac{1}{12}z^{-3}}
\]

\[
= \frac{3 + (-\frac{5}{4} - \frac{1}{4} + \frac{7}{12})z^{-1} + \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{12}\right)z^{-2}}{1 - 2z^{-1} + \frac{8}{12}z^{-2} - \frac{1}{12}z^{-3}}
\]

\[
= \frac{3 + \left(-\frac{3}{2} + \frac{7}{12}\right)z^{-1} + \left(-\frac{3}{12}\right)z^{-2}}{1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}
\]

\[
= \frac{3 - \frac{19}{6}z^{-1} + \frac{2}{3}z^{-2}}{(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2})(1 - z^{-1})}
\]

\( x[n] = u[n] \)

\( X(z) = \frac{1}{1 - z^{-1}} \)

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{3 - \frac{19}{6}z^{-1} + \frac{2}{3}z^{-2}}{1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}
\]

\[
\Rightarrow y(z) = \frac{7}{12}z^{-1}y(z) + \frac{1}{12}z^{-2}y(z) = 3x(z) - \frac{19}{6}z^{-1}x(z) + \frac{2}{3}z^{-2}x(z)
\]

\[
\Rightarrow y[n] = \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = 3x[n] - \frac{19}{6}x[n-1] + \frac{2}{3}x[n-2]
\]

This is the difference equation.
b) \( H(z) = \frac{1}{1 - \frac{1}{3}z^{-1} + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-1}} \)

\[ b) \quad H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{3}z^{-1} + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-1}} + 1 \]

\[ = \frac{1 - \frac{1}{3}z^{-1} - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}} + \frac{1 - \frac{1}{4}z^{-1} - \frac{1}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + 1 \]

\[ = 1 - \frac{\frac{1}{3}z^{-1} - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}} + 1 - \frac{\frac{1}{4}z^{-1} + \frac{1}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + 1 \]

\[ = 3 - \frac{2}{3} \frac{z^{-1}}{1 - \frac{1}{3}z^{-1}} - \frac{1}{2} \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} \]

\[ h[n] = 3 \delta[n] - \frac{2}{3} \left( \frac{1}{3} \right)^{n-1} u[n-1] - \frac{1}{2} \left( \frac{1}{4} \right)^{n-1} u[n-1] \]

c) The system has poles at \( z = \frac{1}{3} \) and \( z = \frac{1}{4} \). As the system is causal, the ROC would be \( |z| > \frac{1}{2} \) which includes the unit circle. So the system is stable.
The system in the figure has a system function $H(z)$ with all the poles and zeros inside the unit circle. As the system is causal and the ROC $|z| > r_{\text{max}}$ includes the unit circle, the system is stable.

Inverses system is such a system when cascaded with $H(z)$, the overall effective system function will be unity:

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

For $H(z) = \frac{b_0}{a_0} \sum_{k=1}^{M} \frac{1 - c_k z^{-1}}{\sum_{\nu=1}^{N} (1 - d_{\nu} z^{-1})}$

$$H_i(z) = \frac{a_0}{b_0} \sum_{k=1}^{N} \frac{1 - d_k z^{-1}}{\sum_{\nu=1}^{M} (1 - c_{\nu} z^{-1})}$$

The poles of $H(z)$ are going to the zeros of $H_i(z)$ and vice versa.

From the figure, we can see that all the poles and zeros are inside the unit circle. After inversion, the poles and zeros of the inverse system will be inside the unit circle. As the ROC of the given system includes unit circle because of the causality with appropriate...
Inverse system should have ROC that overlaps the ROC of the original system. That is only possible if the inverse system is causal as well. All the poles of the inverse system are inside the unit circle, the system would be stable.
\( \text{\# } H_1(e^{jw}) = e^{jw} + e^{-jw} = 2 \cos 2w \)

\( \forall H_1(e^{jw}) = -2w \)

\( \text{and } (H_1(e^{jw})) = -\frac{d}{dw}(-2w) = 2 \)


\( \text{\# } H_2(e^{jw}) = e^{jw} + 1 + e^{-jw} + e^{-2jw} + e^{-3jw} - e^{-j4w} \)

\( H_2(e^{jw}) = e^{j\frac{3\omega}{2}} \left[ -e^{j\frac{5\omega}{2}} + e^{j\frac{3\omega}{2}} + e^{j\frac{3\omega}{2}} + e^{-j2\omega} + e^{-j3\omega} - e^{-j5\omega} \right] \)

\( \forall H_2(e^{jw}) = -\frac{3\omega}{2} \)

\( \text{and } (H_2(e^{jw})) = -\frac{d}{dw}(-\frac{3\omega}{2}) = \frac{3}{2} \)

\( \text{\# } H_3(n) = S[n-1] - S[n-3] \)

\( \text{\# } H_3(e^{jw}) = e^{-jw} - e^{-j3w} = e^{-j2w} \left( e^{jw} - e^{-jw} \right) = 2j\sin \omega e^{-j2w} \)

\( \forall H_3(e^{jw}) = \frac{\pi}{2} - 2w \)

\( \text{and } (H_3(e^{jw})) = -\frac{d}{dw}[\frac{\pi}{2} - 2w] = 2 \)

\( h_4(e^{jw}) = e^{jw} + 3e^{-jw} - e^{-j2w} + 2e^{-j3w} - e^{-j4w} + 3e^{-j5w} + e^{-j7w} \)

\[
= e^{-j3w} \left[ e^{j4w} + 3e^{-j2w} - e^{j5w} \right] + 2 - e^{-jw} + 3e^{-j2w} + e^{-j4w} \\
= e^{-j3w} \left[ \cos 4w + 6 \cos 2w - 2 \cos w + 2 \right]
\]

\( \frac{d}{dw} \left( h_4(e^{jw}) \right) = -3w \)

\( q_{nd} \left( H_4(e^{jw}) \right) = -\frac{d}{dw} (-3w) = 3 \)

\( h_5[n] = -8[n-1] + 2 \delta[n-2] - 8 \delta[n-4] + 8 \delta[n-5] \)

\( h_5(e^{jw}) = -e^{jw} + 2e^{-j2w} - 2e^{-j4w} + e^{-j5w} \)

\[
= e^{-j3w} \left[ -e^{j2w} + 2e^{jw} - 2e^{-jw} + e^{-j2w} \right] \\
= e^{-j3w} \left[ -2e^{j5w} + 4e^{j3w} \right] \\
= 2e^{-j3w} \left[ 2 \sin w - \sin 2w \right] = 2e^{-j3w} \left[ 2 \sin w - \sin 2w \right]
\]

\( \frac{d}{dw} \left( h_5(e^{jw}) \right) = \pi_2 - 3w \)

\( q_{nd} \left( H_5(e^{jw}) \right) = -\frac{d}{dw} (\pi_2 - 3w) = 3 \)


\( h_6(e^{jw}) = 1 + e^{-jw} + e^{-j2w} + e^{-j3w} + e^{-j4w} + e^{-j5w} + e^{-j6w} + e^{-j7w} \)

\[
= e^{-j3w/2} \left[ e^{j3w/2} + e^{j5w/2} + e^{j7w/2} + e^{jw/2} + e^{-jw/2} + e^{-j5w/2} + e^{-j7w/2} \right] \\
= e^{-j5w/2} + e^{-j7w/2} \\
+ \cos 7w/2 + \cos 5w/2 + \cos 3w/2 + \cos w/2
\]

\( \frac{d}{dw} \left( h_6(e^{jw}) \right) = -7w/2 \)

\( q_{nd} \left( H_6(e^{jw}) \right) = -\frac{d}{dw} (-7w/2) = 7/2 \)
From the Fourier transform of the input $x[n]$ we can see that it has frequency components (narrowband) around $\omega = 0.125$ and $\omega = 0.5$.

Also the discrete-time signal shows that the lowest frequency component arrives first followed by the highest frequency component. These two are followed by the middle frequency component.

From the frequency response magnitude of the filter $H$, we can see that the frequency component around $\omega = 0.125$ will have a gain of around 1.8 from the filter and would experience a group delay of around 40 samples. So the envelope center of the narrowband signal around $\omega = 0.125$ gets delayed by 40 sample to around 640 samples.

The narrowband signal with center frequency around $\omega = 0.3$ will have a gain around 1.5 and will experience a delay of 80 samples. This causes the center of the envelope to move to around 180 samples.

Because the filter is a stop band filter with notch around $\omega = 0.5$, most of the signal gets attenuated.

Moreover, the group delay does not add very significant delay to the narrowband signal with center frequency $\omega = 0.5$. 
The output $y_{2[M]}$ shows the same effect as we got from the analysis.