Strictly Stationary Random Sequence: $X_n$

$$F(X_n, X_{n+1}, \ldots, X_{n+m}) =$$

$$F(X_{n+k}, X_{n+1+k}, \ldots, X_{n+m+k})$$

forall $k, m, n$.

* Wide sense stationary sequence, (sometimes it's just called stationary)

- $E[|X[n]|] = \mu$ constant
- $E[X_{[n+k]}X_{[n+l]}] = K_{xx}[k+n,l+n]$ if $k, l \geq 0$
- $K_{xx}[k,l] = K_{xx}[k-l]$ if $k, l \geq 0$
- $\text{Var}(X_{[n]}) = E[X_{[n]}^2] = E[X_{[n]}^2] = \sigma^2 < \infty$
ex: \( X[n] = A \) \( \sim U(2, 5) \)

\[
P(X[n] = x_1, X[n+b] = x_2) = ? ? ?
\]

Strictly stationary sequence

strictly stationary \( \Rightarrow \) stationary (wide sense)

ex: \( X[n] = nA \)

Is this sequence strictly stationary?

No!

\[
P(X[n] < a) \neq P(X[n+b] < a)
\]
White noise

\[ E\left[ X[n] \right] = 0 \]
\[ \text{Var} \left[ X[n] \right] = \sigma^2 < \infty \]
\[ K_{xx}[k, l] = K_{xx}[k-l] \]

ex: let \( X[n] \) be i.i.d.
\[ X[n] \sim N(0, 1) \]
\( X[n] \) is white noise.

ex: \( X[n] = \begin{cases} \sum \left( Y[n] \right) & \text{if } n \text{ is even} \\
\frac{1}{\sqrt{2}} \left( Y[n] - 1 \right) & \text{if } n \text{ is odd} \end{cases} \)

\( Y[n] \) is i.i.d. \( Y[n] \sim N(0, 1) \)

is \( X[n] \) stationary?
\[ E \{ X[n] \} = 0 \]

why?
\[
\begin{align*}
\text{n is even: } & \quad E \{ X[n] \} = E \{ Y[n] \} = 0 \\
\text{n is odd: } & \quad E \{ X[n] \} = \frac{E \{ Y[n] \} - 1}{\sqrt{2}} \\
& \quad \frac{E - 1}{\sqrt{2}} = 0
\end{align*}
\]

\[ \Var \{ X[n] \} : \]
\[
\begin{align*}
\text{n is even: } & \quad \Var \{ X[n] \} = 1 \\
\text{n is odd: } & \quad \Var \{ X[n] \} \\
& \quad = \frac{\Var \{ Y[n] \} - 1}{2}
\end{align*}
\]

why?
\[
\text{Cov} : \quad K_{XX} \{ n, n-k \} = 0
\]
became
\[
\text{Y[n] is iid } \Rightarrow \text{X[n] is iid}.
\]
X[n] is strictly stationary?

No!

\[ P(X[n] \leq 0) = P(Y[n] \leq 0) = 0.5 \]

for n even

\[ P(X[m] \leq 0 = P\left(\frac{1}{\sqrt{2}} X[m] - 1 \leq 0\right) \]

\[ = 0.6 \sim \ldots \]

n odd.

\[ P(X[n] \leq 0) \neq P(X[m] \leq 0) \]

m \neq n.
Wide sense Gaussian random sequence also implies that it is a strictly stationary.

General case:

$$K = \begin{bmatrix} X_1 & X_2 & \cdots & X_m \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \\ \vdots & \vdots \\ K_{m1} & K_{m2} \end{bmatrix}$$

For wide sense stationary:

$$K_{XX} = \begin{bmatrix} b_1 & b_2 & b_2 & b_2 \\ b_2 & b_1 & b_2 & b_2 \\ b_2 & b_2 & b_1 & b_2 \\ b_2 & b_2 & b_2 & b_1 \end{bmatrix}$$

Wide sense Gaussian random sequence $\Rightarrow$ strictly stationary because the joint distribution is completely specified by first and second order moments.
let \[ X[n] = \cos n\theta \]

\[ W \sim U(0, 2\pi) \]

\[ E[X[n]] = \frac{1}{2\pi} \int_{0}^{2\pi} \cos n\theta \, d\theta = 0 \]

\[ \text{Var}[X[n]] = \frac{1}{2\pi} \int_{0}^{2\pi} \cos^2 n\theta \, d\theta \]

\[ = \frac{1}{2\pi} \left( \int_{0}^{2\pi} (1 + \cos 2n\theta) \, d\theta \right) \]

\[ = \frac{1}{2} + 0 \]

Cov: \[ K_{XX} \]

\[ K_{XX}[n, n-k] = \frac{1}{2\pi} \int_{0}^{2\pi} \cos n\theta \cos (n-k)\theta \, d\theta \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{\cos k\theta + \cos (2n-k)\theta}{2} \right) \, d\theta \]

\[ = 0 \quad \text{for} \quad k \neq 2n \]
In the past, you learn LTI system:

If the system is not time invariant, but is linear, then:
\( Y[n] = \sum_{k=-\infty}^{\infty} h[n-k] X[k] \)

\( h[n, k] \) is general form for impulse response for system that is not TI.

\( R_{xy}[m,n] = E[X[m] Y[n]^*] = E\left[ X[m] \sum_{k=-\infty}^{\infty} h[n-k]^* X[k] \right] = \sum_{k=-\infty}^{\infty} h[n-k]^* E[X[m] X[k]] = \sum_{k=-\infty}^{\infty} h[n-k] R_{xx}[m,k] \)
\[ R_{yy}[m,n] = E\left[ X[m]X[n]\right] \]

\[ = E\left[ \sum_{k=-\infty}^{\infty} h[m,k]X[k] \sum_{j=-\infty}^{\infty} h[m,j]X[j]\right] \]

\[ = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[m,k]h[m,j] E[X[k]X[j]] \]

\[ = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[m,k]h[m,j] R_{xx}[k,j] \]

Now, we will study WSS signals. But first, let's consider the properties of \( R_{xx} \) of WSS signal.

**Properties of \( R_{xx}[m] \)**

1. \( |R_{xx}[m]| \leq R_{xx}[0] + \rho \)
2. \( R_{xx}[0] \geq 0 \)
3. \( R_{xx}[m] = R_{xx}[-m] \)
4. For \( N > 0 \), and all complex numbers \( a_1, a_2, \ldots, a_N \), we have

\[ \sum_{n=1}^{N} \sum_{k=1}^{N} a_n a_k R_{xx}[k-n] \geq 0 \]
Proof: For convenience, assume that $X[n]$ is real.

1. $E[(X[m] - X[0])^2] \geq 0$

$\Rightarrow E[X[m]^2 - 2X[m]X[0] + X[0]^2]$

$= E[X[m]^2] - 2R_{XX}[m] + E[X[0]^2] \geq 0$

$\Rightarrow E[X[0]^2] \geq R_{XX}[m]$

$\Rightarrow R_{XX}[0]$

2. $R_{XX}[0] = E[X[m]^2] \geq 0$

3. $R_{XX}[m] = E[X[m+\ell] \times [\ell]]$

$= E[X[\ell] \times [\ell-m]]$

$= E[X[\ell] \times [\ell-m]]$

$= \delta_{\ell-m} R_{XX}[-m]$