We will consider LTI system only. If you are interested in L system, please read the book or consult the.

Assume \( x[n] \) is W.S.S.

1) Mean of \( Y[n] \)

\[
M_y = E[Y[n]] = E\left[ \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right] = \sum_{k=-\infty}^{\infty} h[k] E[x[n-k]]
\]

2) \( R_{xy}[n+k,n] \)

\[
R_{xy}[n+k,n] = E\left[ X[n+k] Y[n] \right] = E\left[ X[n+k] \sum_{i=-\infty}^{\infty} x[i] h[n-i] \right] = \sum_{i=-\infty}^{\infty} E\left[ X[n+k] x[i] \right] h[n-i]
\]

Let \( j = i - n \)

\[
= \sum_{i=-\infty}^{\infty} R_{xx}[k-j] h^x[-j]
\]
\[ R_{yy}[n+k, n] = E\left[ y[n+k], y[n] \right] = E\left[ \sum_{m=-\infty}^{\infty} h[m] x[n+m-k] y[n-m] \right] = \sum_{m=-\infty}^{\infty} h[m] E\left[ x[n+m-k] y[n-m] \right] = \sum_{m=-\infty}^{\infty} h[m] R_{xy}[k-m] = h[k] \ast R_{xy}[k] = h[k] \ast h[-k] \ast R_{xx}[k] = \delta[k] \circledast R_{xx}[k] \]
\[ Y[n] = X[n] - X[n-1] \]

\( X[n] \) is W.S.S. with \( R_{XX}[k] = e^{\alpha k} \)

Compute \( R_{YY}[n] \).

\[ h[n] = S[n] - S[n-1] \]

\[ g[n] = h[n] \ast h[-n] \]

\[ = (S[n] - S[n-1]) \ast (S[n] - S[n+1]) \]


\(( S[n-k] \ast x[n] = x[n-k] ) \)

\[ = 2S[n] - S[n+1] - S[n-1] \]

\[ R_{YY}[n] = g[n] \ast R_{XX}[n] \]

\[ = (2S[n] - S[n+1] - S[n-1]) \ast e^{\alpha n} \]

\[ = 2a^{n}[n] - a^{n+1}[n] - a^{n-1}[n] \]
\[ a = \frac{1}{2} \]

\[ n = 1 \cdot 2 \left( \frac{1}{2} \right)^1 - \left( \frac{1}{2} \right)^2 = 1 - \frac{1}{4} - 1 = -\frac{1}{4} \]

**Power spectral density**

**Definition:**

\[
S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}[n] e^{-j\omega n}
\]

\[
R_{xx}[n] = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega n} d\omega
\]
1) \[ S_{yy}(\omega) = H(\omega) H^*(\omega) S_{xx}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \]

2) \[ E[X[n]X^[\ast][n]] = R_{xx}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}(\omega) d\omega \]

Suppose \( H(\omega) = \sum_{n=-\infty}^{\infty} 1 \) for \( |\omega - \omega_0| < \delta \)

\[ R_{yy}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{yy}(\omega) d\omega \]
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n} 1^2 H(w) \cdot S_{xx}(w) \, dw \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}(w) \, dw \]

\[ = \frac{1}{2\pi} \int_{w_0 - \delta}^{w_0 + \delta} S_{xx}(w) \, dw \]

\[ \frac{2\pi S_{xx}(w)}{2\pi} \]

\[ \frac{2\pi S_{xx}(w)}{2\pi} = \frac{2\pi S_{xx}(w)}{2\pi} \]

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\[ = \frac{2\pi S_{xx}(w)}{2\pi} \]

- \( S(w) \) is real.

(because \( S(w) \) is FT of \( R_{xx}[n] \) that is conjugate symmetric)

- If \( x[n] \) is real, then \( S_{xx}(w) \) is an even function.

(Proprieties of FT)

- \( S(w) \geq 0 \) for \( w \).
Why we should view FT of $R_{xx}[n]$ as power spectral density?

$X_N[n] = W_N[n] X[n]$

$W_N[n] = \frac{1}{N} \sum_{0}^{N-1} e^{-j\frac{2\pi}{N} kn}$

Windowing function:

$\frac{1}{2N+1} E \left[ |X_N(w)|^2 \right] = \frac{1}{2N+1} \sum_{k=-N}^{N} \sum_{l=-N}^{N} X[k] X[l] e^{-j2\pi(k-l)w} e^{-j2\pi kl}$

$= \frac{1}{2N+1} \sum_{k=-N}^{N} \sum_{l=-N}^{N} R_{xx}[k-l] e^{-j2\pi kl}$

let $m = k-l$

$= \frac{1}{2N+1} \sum_{m=-2N}^{2N} R_{xx}[m] e^{-j2\pi mw}$

$= \frac{1}{2N+1} \sum_{m=-2N}^{2N} (2N+1) R_{xx}[m] e^{-j2\pi mw}$

$= \frac{1}{2N+1} \sum_{m=-2N}^{2N} R_{xx}[m] - 1 |m|
\[
\frac{1}{2N+1} \sum_{m=-2N}^{2N} R_{xx}[m] (2N+1 - m) e^{-j\omega m}
\]

\[
\leq \frac{1}{2N+1} \sum_{m=-2N}^{2N} \left| R_{xx}[m] \right| (2N+1) e^{-j\omega m} \sim 1 \sum_{m=-2N}^{2N} R_{xx}[m] e^{-j\omega m}
\]

\[
\leq 2N \sum_{m=-2N}^{2N} R_{xx}[m] e^{-j\omega m} - \frac{1}{2N+1} \sum_{m=-2N}^{2N} (m) R_{xx}[m] e^{-j\omega m}
\]

Now, \( R_{xx}[m] \) decreases sufficiently quickly then \( A \to 0 \) as \( N \to \infty \).

\[
\mathbb{E} \left[ X_N^2(\omega) \right] \quad N \to \infty \quad \sum_{m=-\infty}^{\infty} R_{xx}[m] e^{-j\omega m}
\]
Wiener's Filter.

\[ x[n] \xrightarrow{h[n]} y[n] \xrightarrow{+} e[n] \]

Suppose \( h[n] \) is linear, and the goal is to minimize the mean square error between \( y[n] \) and \( d[n] \). \( d[n] \) is given:

- \( d[n] \) is the desired signal
- \( e[n] \) is the error signal
- \( y[n] \) is the estimated \( d[n] \)
- For Wiener's filter,

\[ y[n] = w_{0}j x[n] + w_{1}j x[n-1] + \ldots + w_{N}j x[n-N] \]

where \( w_{0}, w_{1}, \ldots, w_{N} \) are the Wiener's filter coefficients with \( N+1 \) taps.
We want to minimize
\[ E[\epsilon^2] = E[(y[n] - d[n])^2] \]

Now,
\[ y[n] = \sum_{i=0}^{k} w[i] x[n-i] \]

\[ \frac{d}{d w[i]} E[\epsilon^2] = \frac{d}{d w[i]} E[(y[n] - d[n])^2] \]
\[ = \frac{d}{d w[i]} E \left[ \left( \sum_{j=0}^{k} w[j] x[n-j] - d[n] \right) x[n-i] \right] = 0 \]

\[ \Rightarrow \sum_{j=0}^{k} E \left[ x[n-j] x[n-i] w[j] \right] = E \left[ d[n] x[n-i] \right] \]

Hence,
\[ R_{xx}[i-j] w[j] = R_{dx}[i] \quad i = 1, \ldots, k \]

In matrix form,
\[ \begin{bmatrix} R_{xx}[0] & R_{xx}[1] & \cdots & R_{xx}[k] \\ R_{xx}[1] & R_{xx}[0] \\ \vdots & \ddots & \ddots & \ddots \\ R_{xx}[k] & \cdots & R_{xx}[0] & R_{xx}[k] \end{bmatrix} \begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[k] \end{bmatrix} = \begin{bmatrix} R_{dx}[0] \\ R_{dx}[1] \\ \vdots \\ R_{dx}[k] \end{bmatrix} \]
Application:

Predict \( x[n+1] \) based on \( x[n+2], x[n+1], x[n-1], x[n-2] \).