Markov Random Sequence

\[
\mathbb{P}(x_{n+k} \mid x_n, x_{n-1}, \ldots, x_0) = \mathbb{P}(x_{n+k} \mid x_n)
\]

\[
P_x(x_{n+k} \mid x_n, x_{n-1}, \ldots, x_0) = P_x(x_{n+k} \mid x_n)
\]

Typical setting: \(k = 1\)

Noted that it is often easier to compute the joint or the marginal of the random Markov sequence.

\[
\mathbb{P}(x_0, x_1, \ldots, x_n) = \mathbb{P}(x_0) \mathbb{P}(x_1, x_2, \ldots, x_n \mid x_0)
\]

\[
= \mathbb{P}(x_0) \mathbb{P}(x_1, x_2, \ldots, x_n \mid x_1, x_0)
\]

\[
= \mathbb{P}(x_0) \mathbb{P}(x_1 \mid x_0) \mathbb{P}(x_2, x_3, \ldots, x_n \mid x_2, x_1)
\]

(Markov property)

: continue to expand this term

we obtain

\[
\mathbb{P}(x_0, x_2, \ldots, x_n) = \mathbb{P}(x_0) \mathbb{P}(x_2 \mid x_0) \mathbb{P}(x_3 \mid x_2) \ldots
\]

\[
= \mathbb{P}(x_0) \prod_{i=1}^{n} \mathbb{P}(x_i \mid x_{i-1})
\]
\( X_0 \) be a random Markov sequence \( n \geq 0 \)

\[ f(x) = \mathcal{N}(0, \sigma^2), \quad \text{and} \]

\[ f(x_n | x_{n-1}) = \mathcal{N}(p x_{n-1}, \sigma^2) \]

Suppose we want to compute marginal

\[ f_{X_n}(x) \]

\[ f_{X_n}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_X(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_{n-1} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_X(x_1) f_X(x_2 | x_1) \cdots f_X(x_n | x_{n-1}) \, dx_1 \, dx_2 \cdots dx_n \]

It is easier to do compute

if \( f_X(x_1, x_2, \ldots, x_n) \) is product of separable variables.

Suppose \( M_{X \mid \mathcal{E}} \approx E[X | \mathcal{E}] \approx x_n \), we want to compute \( M_{X \mid \mathcal{E}} \).
\[ E[X[n]] = E[E[X[n]]] \]

\[ E[X[n]] = E[X[n-1]] \]

\[ = \rho E[X[n-2]] \]

\[ = \rho E[X[n-3]] \]

\[ \vdots \]

\[ = \rho^n E[X[0]] = 0. \]

\[ E[X^2[n]] = E[E[X^2[n] | X[n-1]]] \]

\[ = E[\beta^2 + \rho^2 X^2[n-1]] \]

\[ = \beta^2 + \rho^2 E[X^2[n-1]] \]

\[ = \beta^2 + \rho^2(\beta^2 + \rho^2 E[X^2[n-2]]) \]

\[ = \beta^2 + \rho^2\beta^2 + \rho^4 E[X^2[n-2]] \]

\[ \vdots \]

\[ = \beta^2 (1 + \rho^2 + \rho^4 + \ldots + \rho^{2(n-1)}) \]

\[ \rho \text{ is } | \rho | < 1 \]

\[ \Rightarrow E[X^2[n]] \xrightarrow{n \to \infty} \frac{\beta^2}{1 - \rho^2} \]
**Markov Chain**

**Definition:** A discrete-time Markov chain is a random sequence \( X(n) \) whose \( N \)th order conditional probability satisfies
\[
P_X(x_n | x_{n-1}, \ldots, x_{n-N}) = P_X(x_n | x_{n-1}) \quad \forall n, \quad N > 1
\]

Typically, we represent \( X(n) \) by a diagram or transition probability matrix.

\[
P = \begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}
\]

Note that
Irreducible Markov chain:
Every state can be reached by any other state with non-zero probability.

Every state can reach others.
A periodic:  \[ \overbrace{1,1,2,1,1,2,1,1}^{n} \to \overbrace{1,2,2,1,2,1,2,1}^{n} \]

Main result: If a Markov chain is irreducible and aperiodic, then there exists a unique stationary distribution.

\[ \pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_m \end{bmatrix} \text{ such that } \]

\[ \pi^T P^n \overset{n \to \infty}{\to} \pi^T \text{ for any initial distribution.} \]

transition probability matrix \[ \pi \]
Example: Suppose that the probability that it will rain tomorrow given it rains today is 0.9 and the probability that it will not rain tomorrow given that it does not rain today is 0.8.

Find the probability that it will rain in 10 days, given that it rain today.

Initial distribution

\[ y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ y_{10}^T = y_0^T P = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \]

\[ = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} \]

\( y_{10} \) = [0.9, 0.1]

- Rain tomorrow
- No rain tomorrow
\[ \lim_{n \to \infty} \pi^T P^n = \pi \]

Stationary probability

regardless of initial \( \pi_0 \)

Ex: DMV

Number of people waiting in line at

station.

\[ P = \begin{bmatrix}
\pi_0 & \pi_1 & 0 & 0 & \cdots \\
\pi_0 & \pi_1 & \pi_2 & 0 & \cdots \\
\pi_0 & \pi_1 & \pi_2 & \pi_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix} \]
Method for finding \( \Pi \):

\[
P^T \Pi = \Pi^T
\]

(1)

\[
\sum_{i} \Pi_i = 1
\]

(2)

\[
\Pi = \begin{bmatrix}
0.0 & 1 \\
0.2 & 0.8
\end{bmatrix}
\]

From (1):

\[
\begin{bmatrix}
\Pi_1 \\
\Pi_2
\end{bmatrix} \begin{bmatrix}
0.9 \\
0.2
\end{bmatrix} = \begin{bmatrix}
\Pi_1 \\
\Pi_2
\end{bmatrix}
\]

\[
(0.9)\Pi_1 + (0.2)\Pi_2 = \Pi_1
\]

From (2):

\[
\Pi_1 + \Pi_2 = 1
\]

\[
\Pi_1 = \frac{2}{3}
\]

\[
\Pi_2 = \frac{1}{3}
\]
\[ X[n] \rightarrow X(t) \]

\[ M_x(t) = E[X(t)] \]

\[ R_{xx}(t_1, t_2) = E[X(t_1)X^*(t_2)] \]

**Ex:** a) Consider a random process whose \( A \) and \( \theta \) are independent r.v.

\[ \theta \sim U[-\pi, \pi] \]

\[ X(t) = A \sin (w_0 t + \theta) \]

\[ E[X(t)] = E[A \sin (w_0 t + \theta)] \]

\[ = E[A] E[\sin (w_0 t + \theta)] \]

\[ = E[A] \int_{-\pi}^{\pi} \sin (w_0 t + \theta) \frac{1}{2\pi} \, d\theta = 0 \]
b) \[ \mathbb{E}[X(t_1)X(t_2)] = \mathbb{E}[X(t_1)] \mathbb{E}[X(t_2)] = \mathbb{E}[X(t_1)] \mathbb{E}[X(t_2)] = \mathbb{E}[A^2] \mathbb{E}[\sin(w_0 t_1 + \theta) \sin(w_0 t_2 + \theta)] \\
= \mathbb{E}[A^2] \mathbb{E}[\frac{1}{2} \cos(w_0 (t_1 - t_2)) - \frac{1}{2} \cos(w_0 (t_1 + t_2) + 2\theta)] \\
= \mathbb{E}[A^2] \frac{\cos(w_0 (t_1 - t_2))}{2} \\
= \mathbb{E}[A^2] \frac{\cos(w_0 (t_1 + t_2) + 2\theta)}{2} \\
\text{Note:} \quad \mathbb{E}[\cos(w_0 (t_1 + t_2) + 2\theta)] = 0 \\

W.S.S \\

\[ M_x(t) = \text{constant} \]
\[ R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2) = R_{XX}(t) \]
\[ R_{XX}[0] < \infty \]