1. First, do $z$-transform to $y[n-1] + \frac{1}{3} y[n-2] = x[n]$.

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} + \frac{1}{3} z^{-2}} = \frac{z}{1 + \frac{1}{3} z^{-1}} \]

- If $|z| < \frac{1}{3}$, then do inverse $z$-transform to $H(z)$.

\[ h[n] = -\left(\frac{1}{3}\right)^{n+1} u[-(n+1) - 1] = -\left(\frac{1}{3}\right)^{n+1} u[-n-2] \]

\[ = \frac{1}{3} \left(\frac{1}{3}\right)^n u[-n-2] \]

- If $|z| > \frac{1}{3}$, do inverse $z$-transform to $H(z)$.

\[ h[n] = \left(\frac{1}{3}\right)^{n+1} u[n+1] \]

Therefore, (a) and (d) are correct!

2. (a) $X(z) = -\frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3} z^{-1}} + \frac{4}{3} \cdot \frac{1}{1 - 2 z^{-1}}$

\[ = \frac{-\frac{1}{3} (1 - 2 z^{-1}) + \frac{4}{3} (1 - \frac{1}{3} z^{-1})}{(1 - \frac{1}{3} z^{-1})(1 - 2 z^{-1})} \]

\[ = \frac{1}{(1 - \frac{1}{3} z^{-1})(1 - 2 z^{-1})} \quad \frac{1}{2} < |z| < 2 \]

(b) As we can see the poles of $X(z)$ is $\frac{1}{3}$ and $2$. The poles of $Y(z)$ also be $\frac{1}{3}$ and $2$. Hence, the ROC of $Y(z)$ is still $\frac{1}{2} < |z| < 2$.

(c) $H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2}$

Inverse $z$-transform:

\[ h[n] = \delta[n] - \delta[n-2] \]
3. In order to solve this problem, we want to talk about the relationship between symmetry and group delay.

Suppose we have an impulse response like this:

\[ h[n] \]

The symmetry of this impulse response is \( \frac{n_0}{2} \). Then we can write down the Fourier transform of the impulse response \( h[n] \):

\[
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[m-n] e^{-j\omega n} \quad (m = 2, \frac{n_0}{2} = n_0)
\]

Using \( l \) to replace \( m-n \), \( n = m-l \)

\[
H(e^{j\omega}) = \sum_{l=-\infty}^{\infty} h[l] e^{-j\omega (m-l)}
\]

\[
= e^{-j\omega m} \sum_{l=-\infty}^{\infty} h[l] e^{j\omega l}
\]

\[
= e^{-j\omega m} H(e^{-j\omega})
\]

Then, let's talk about the relationship between \( H(e^{j\omega}) \) and \( H(e^{-j\omega}) \).

\[
\begin{align*}
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \quad \text{(Definition)} \\
\quad e^{-j\omega n} &= \cos n\omega - j\sin n\omega \\
\Rightarrow H(e^{j\omega}) &= \text{Re}(H(e^{j\omega})) + j\text{Im}(H(e^{j\omega})) \\
&= \sum_{n=-\infty}^{\infty} h[n] \cos n\omega - j\sum_{n=-\infty}^{\infty} h[n] \sin n\omega
\end{align*}
\]
In the same way,
\[ H(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cos(n\omega) + j\sum_{n=-\infty}^{\infty} h[n] \sin(n\omega) \] (3)
From (2) and (3), we know \[ H(e^{j\omega}) = H(e^{-j\omega}) \] (4).

From (1) and (4), we can have
\[ H(e^{j\omega}) = e^{-j\omega m} H^*(e^{j\omega}) \] (5)
We can use amplitude and phase术语 to express \[ H(e^{j\omega}) \] and \[ H^*(e^{j\omega}) \].
\[ H(e^{j\omega}) = A e^{j\phi(\omega)} \] (6)
\[ H^*(e^{j\omega}) = A e^{-j\phi(\omega)} \] (7)

From (5), (6), (7)
\[ H(e^{j\omega}) = A e^{j\phi(\omega)} = e^{-j\omega m} A e^{-j\phi(\omega)} \]
\[ \Rightarrow e^{j2\phi(\omega)} = e^{-j\omega m} \]
\[ \Rightarrow 2\phi(\omega) = -\omega m \]
\[ \Rightarrow \phi(\omega) = \frac{-\omega m}{2} + 2\pi k, \quad k \in \mathbb{Z} \]
\[ \text{gcd} (H(e^{j\omega})) = -\frac{d\phi(\omega)}{d\omega} = \frac{m}{2} \quad \Rightarrow \quad \frac{m}{2} = \frac{n_0}{2} \]

Hence, we can get the conclusion: The symmetry \[ \frac{n_0}{2} \] is the good group delay.

1. Group delay is 2.
2. Group delay is 1.5.
3. Group delay is 2
4. Group delay is 3
(5) Group delay is 3.

(6) Group delay is 3.5.

You can also calculate it. Let's take $h[n]$ for example.

$$H_1(e^{j\omega}) = 1 + e^{-j4\omega}$$

$$= e^{-j2\omega} (e^{j2\omega} + e^{-j2\omega})$$

$$= 2e^{-j2\omega} \cos 2\omega$$

$$\text{Arg}[H_1(e^{j\omega})] = -2\omega$$

$$\text{gcd} \left( \omega \rightarrow H_1(e^{j\omega}) \right) = -\frac{d\text{Arg}[H_1(e^{j\omega})]}{d\omega} = 2$$

4.

As we can see from figure 3.

The narrow band signal $\omega = 0.12\pi$ has a gain equal to $1.8$, and the group delay of it is 40 samples.

The narrow band signal $\omega = 0.3\pi$ has a gain around 1.5, and the group delay is 80 samples.

The narrow band signal $\omega = 0.5\pi$ has a gain equal to 0, that means this signal disappears after filter A. The group delay is nearly to 0.

$\Rightarrow$ In a word, only $y_2[n]$ is Figure 4 can be the solution!