Random Vectors and Estimation

1. Problem 5.29
2. Problem 5.30
3. Problem 5.31
4. Problem 6.3
5. Problem 6.26

6. An experimental trial produces random variables $X_1$ and $X_2$ with correlation $r = E[X_1X_2]$. To estimate $r$, we perform $n$ independent trials and form the estimate:

$$
\hat{R}_n = \frac{1}{n} \sum_{i=1}^{n} X_1(i)X_2(i), \tag{1}
$$

where $X_1(i)$ and $X_2(i)$ are samples of $X_1$ and $X_2$ on trial $i$. Show that if $\text{Var}[X_1X_2]$ is finite, then $\hat{R}_1, \hat{R}_2, \ldots$ is an unbiased, consistent sequence of estimates of $r$.

7. Given the i.i.d samples of $X_1, X_2, \ldots$ of $X$, define sequence $Y_1, Y_2, \ldots$ by

$$
Y_k = (X_{2k-1} - \frac{X_{2k-1} + X_{2k}}{2})^2 + (X_{2k} - \frac{X_{2k-1} + X_{2k}}{2})^2. \tag{2}
$$

Show that if $E[X^k] < \infty$ for $k = 1, 2, 3, 4$ then the sample mean $M_n(Y) = \frac{1}{n} \sum_{i=1}^{n} Y_i$ is a consistent, unbiased estimate of $\text{Var}[X]$.

8. We have learned in the class about the law of large number for i.i.d. sequence. In this problem, we develop a weak law of large numbers for correlated sequence $X_1, X_2, \ldots$ of identical random variables. In particular, each $X_i$ has expected value $E[X_i] = \mu$, and the random sequence has covariance function

$$
K_{m,m+k} = \text{Cov}(X_m, X_{m+k}) = \sigma^2 a^{|k|}, \tag{3}
$$

where $a$ is a constant such that $|a| < 1$. For this correlated random sequence, we can define the sample mean of $n$ samples as

$$
Y_n = \frac{X_1 + X_2 + \cdots + X_n}{n} \tag{4}
$$

(a) Show that

$$
\text{Var}[Y_n] \leq \frac{\sigma^2}{n} \left( \frac{1 + a}{1 - a} \right) \tag{5}
$$

(b) Show that for any $c > 0$,

$$
P(|Y_n - \mu| \geq c) \leq \frac{\sigma^2(1 + a)}{n(1 - a)c^2} \tag{6}
$$

(c) Use (b) to write down a weak law of large number for the correlated sequence $X_i$.

9. Let $Y = AX + Z$ where $X$ is the unknown random vector, $Z$ is random noise vector with zero mean, $Y$ is the observation vector, and $A$ is a known matrix. Assume that $K_{XZ} = 0$. Show that the LMMSE for $x$ is:

$$
\hat{x} = K_X A^T (AK_X A^T + K_Z)^{-1} (y - A\mu_X) + \mu_X \tag{7}
$$