Random Sequence

Definition: \( x[n, \omega] \) is a function that is assigned to each outcome \( \omega \) of a sample space \( \Omega \) (\( \omega \in \Omega \)).

\[ X[n] = \text{random variable} \]
Example:

\[ X[n, w] = A(w) \delta[n] \]

\[ S[n] = \frac{1}{n} \]

\[ A(w) \sim \text{Bern}(p) \]

\[ X[n, 0] = \delta[n] = 0 \quad \frac{1}{n} = 0 \]

\[ X[n, 1] = \frac{1}{n} \quad n \geq 1 \]

Now suppose:

\[ X[n] = A[n] \delta[n] \]

\[ A[n] \sim \text{Bern}(p) \]
Definition: \( E[X[n]] = \sum_{i} \int_{-\infty}^{\infty} x f_{X[n]}(x) \, dx \geq \bigwedge_{i} \, x_i \bigwedge_{i} \, X[n] \)

Definition: A random sequence \( X[n] \) is statistically specified if its \( N \)-order cdf is defined for \( N \geq 1 \).
Example: \[ x[n] = \frac{W}{n} \text{ where} \]
\[ W = \sum_{i=2}^{\infty} \left\{ \begin{array}{ll}
1 & \text{with } p \\
-1 & \text{with } 1-p
\end{array} \right. \]
\[ x[1] = \left\{ \begin{array}{ll}
1 & \text{with } p \\
-1 & \text{with } 1-p
\end{array} \right. \\
x[2] = \left\{ \begin{array}{ll}
\frac{1}{2} & \text{with } \frac{1}{2}p \\
-\frac{1}{2} & \text{with } 1-p
\end{array} \right. \]
\[ E[x|\Omega] = (1) p + (-1)(1-p) \]
\[ p - 1 + p = 2p - 1 \]
\[ E[x|x^2] = \frac{1}{2} p + -\frac{1}{2}(1-p) \]
\[ = p - \frac{1}{2} \]
\[ E[x[n]] = \frac{2p - 1}{n} \]
Definition:

\[ R_{xx}[k, l] = \mathbb{E}[X[k] X[l]^*] \]

is the autocorrelation function.

Definition:

\[ K_{xx}[k, l] = \mathbb{E}[X[k] X[l]^*] - \mathbb{E}[X[k]] \mathbb{E}[X[l]^*] \]

is the auto-covariance function.

Definition:

\[ \sigma^2[n] = \mathbb{E}\left[ (X[n] - \mu_n)^2 \right] \]

is the variance function.

\[ K_{xx}[n, n] = K[n] \]

Properties:

\[ R_{xx}[k, l] = R_{xx}[l, k]^* \]
\[ K_{xx}[k, l] < K_{xx}[l, k] \]
Example

\[ X[n] = W[n] + \alpha X[n-1] \]

\[ X[0] = W[0] \]
\[ X[n] = \sum_{k=0}^{n} \alpha^k W[n-k] \]

Let \( \text{Compute } E[X[n]] = E \left[ \sum_{k=0}^{n} \alpha^k W[n-k] \right] \)

\[ = \sum_{k=0}^{n} \alpha^k E[W[n-k]] = \sum_{k=0}^{n} \alpha^k p = p \left( \frac{1 - \alpha^{n+1}}{1 - \alpha} \right) \]
\[ |\alpha| < 1 \rightarrow \text{asymptotic Mean function change} \]
\[ |\alpha| \geq 1 \rightarrow \text{no convergence.} \]

Compute the Variance:

\[
\text{Var}(X[n]) = \text{Var} \left( \sum_{k=0}^{n} \alpha^{n-k} W[k] \right)
\]

\[
= \sum_{k=0}^{n} \text{Var}(\alpha^{n-k} W[k]) \quad \text{because } W[k] \text{ is i.i.d.}
\]

\[
= \sum_{k=0}^{n} \alpha^{2(n-k)} \text{Var}(W[k])
\]

\[
= \sum_{k=0}^{n} \alpha^{2(n-k)} p(1-p) = p(1-p) \sum_{k=0}^{n} \alpha^{2(n-k)}
\]

\[
= p(1-p) \alpha^{2n} \left( \frac{1 - \alpha^{2(n+1)}}{1 - \alpha^2} \right)
\]

\[
= \frac{2p(1-p) \alpha^{2n}}{1 - \alpha^2}
\]

\[
\Rightarrow \alpha^{2n} \frac{2p(1-p)}{1 - \alpha^2}
\]
\[ E \left[ X[n] X[n+1] \right] = \beta \]

\[ E \left[ X[n] \right] = E \left[ W[0] (W[1] + \alpha W[0]) \right] \]

\[ = E \left[ W[0] W[1] + \alpha W[0]^2 \right] \]

\[ = E \left[ W[0] \right] E \left[ W[1] \right] + \alpha E \left[ W[0]^2 \right] \]

\[ = \beta + \alpha \left( \beta + \frac{\beta^2}{\text{var}(W[0]) + E[W[0]^2]} \right) \]

\[ = \beta^2 + \alpha \beta \]

\[ E \left[ X[n] X[n+1] \right] = \beta \]