Independent increment random sequence.

A random sequence is called an independent increment random sequence if for all integers \( n_1 < n_2 < n_3 \ldots < n_N \), the increments \( X[n_1], X[n_2] - X[n_1], X[n_3] - X[n_2], \ldots, X[n_N] - X[n_{N-1}] \), are jointly independent \( \forall N \geq 1 \).

ex: \( W[n] \) is an i.i.d. Is \( W[n] \) an independent increment random sequence?

\[
\begin{align*}
&\vdots \\
Y[n] &= W[n] - W[n-1]
\end{align*}
\]

Answer: No, \( Y[n] \) depends on \( Y[n-1] \) via \( W[n-1] \)

\[
\]

\( \Rightarrow \) dependent
\[ X[n] = \sum_{i=1}^{n} W[i], \quad \text{where } W[i] \text{ is i.i.d.} \]

Is \( X[n] \) an independent increment random sequence?

\[
\begin{align*}
&\vdots \\
X[n] - X[n-1] &= W[n]
\end{align*}
\]

Since \( W[i] \perp W[j] \Rightarrow X[n] \) is independent increment random sequence.

Note: The independent increment random sequence is good in the sense that:
1) We can compute the joint distribution easily.
2) We can also compute the marginal distribution of \( X[n] \) by simply convolving many independent random variables.

(Note: See the Poisson random sequence, we derived the marginal distribution in earlier lecture.)
A strictly stationary random sequence is a random sequence such that the joint CDF of
\[ X[n+1], X[n+2], \ldots, X[n+N-1] \] and the CDF of
\[ X[n+1+k], X[n+2+k], \ldots, X[n+N-1+k] \] are the same, \( \forall N \geq 1, \forall k \).

\[ P\left( X[n+1] \leq x_n, X[n+2] \leq x_{n+1}, \ldots, X[n+N-1] \leq x_{n+N-1} \right) = P\left( X[n+1+k] \leq x_n, X[n+2+k] \leq x_{n+1}, \ldots, X[n+N-1+k] \leq x_{n+N-1+k} \right) \]

\( \forall N \geq 1, \forall k \), \( \forall x_n, x_{n+1}, \ldots, x_{n+N} \)

\[ F(x_n, x_{n+1}, \ldots, x_{n+N}) = F(x_{n+1+k}, x_{n+2+k}, \ldots, x_{n+N-1+k}) \]

Any i.i.d. sequence is a strictly stationary random sequence.

Wide sense stationary.

Wide sense stationary sequences requires

1) mean function \( \mu[n] \) is a constant
2) the covariance function

\[
K_{xx}[k, l] = K_{xx}[k+n, l+n], \quad \forall k, l, n \in \mathbb{Z}
\]

Let \( n = -l \)

\[
K_{xx}[k, l] = K_{xx}[k-l, 0] = K_{xx}[k-l]
\]

\[
K_{xy}[m] = \mathbb{E}[X[k]X^*[k-m]] = \mu_m^2
\]

W.S.S \( \not\Rightarrow \) S.S.S
S.S.S \( \Rightarrow \) W.S.S

\[
R_{xx}[m] = \mathbb{E}[X[k]X^*[k-m]]
\]
White noise:

- \( E[X[n]] = 0 \)
- \( \text{Var}[X[n]] = \sigma^2 < \infty \)
- \( \delta[k-l] = \delta[k-l] \)

ex: \( X[n] = A \)

\( A \sim U(2, 5) \)

\( A = 2 \) w.p. \( \frac{1}{4} \)
\( A = 3 \) w.p. \( \frac{1}{4} \)
\( A = 4 \) w.p. \( \frac{1}{4} \)
\( A = 5 \) w.p. \( \frac{1}{4} \)

WN?

\( p(X[1] = 2, X[7] = 5) = 0 \)
\( p(X[3] = 2, X[8] = 5) = 0 \)
\[ = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \]

\[ P(X[3] = 2, \ X[9] = 2) = \frac{1}{4} \]

\[ X[η] = n \cdot A \]

\[ A \sim \text{U}(2, 5) \]

\[ P(X[1] = 2) = \frac{1}{4} \quad \exists \Rightarrow \neg \exists s, s, s \]

\[ P(X[10] = 2) = 0 \]
Because $\mu$ is not zero for general i.i.d.

ex: $X[n] = \sum \frac{1}{\sqrt{2}} (Y[n] - 1)$ is even

$Y[n]$ is i.i.d. $Y[n] \sim N(0,1)$

Is $X[n]$ strictly stationary?

W.S.S.? W.N.?
WSS?

\[ E[X[n]] \]
- if \( n \) is even
- if \( n \) is odd

\[ E[X[n]] = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{1}{n} - \frac{1}{2} & \text{if } n \text{ is odd} \end{cases} = 0 \]

\[ \text{Var}[X[n]] = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{1}{n} - \frac{1}{2} & \text{if } n \text{ is odd} \end{cases} = \frac{1}{2} \]

\[ Y[n] \sim N(0,1) \]

\[ E[Y^p] = \begin{cases} 1^p (p-1)! & \text{if } p < 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ \text{Cov}: K_{XX}[n,n-k] = c \cdot 2^{n-k} \]

W.S.S \( \sim \) yes

W.N \( \sim \) yes