1. Information Gain/Information Remain: This question is designed for you to develop the intuitions about entropy, joint entropy, conditional entropy, and mutual information. A roulette wheel is subdivided into 38 numbered compartments of various colours. The distribution of the compartments according to the colour is: 2 green, 18 red, and 18 black. The experiment consists of throwing a small ball onto the rotating roulette wheel. The event, that the ball comes to rest in one of the 38 compartments is equally probable for each compartment.

(a) Suppose we are only interested in the colour only, how much information is there?
(b) Suppose we are interested in both colour and number, how much information is there?
(c) How much information on the numbered outcomes remain from knowing their colours?
(d) How much information we gain about the numbered outcomes from knowing their colours?

2. A simple communication: A binary communication system makes use of the symbols 0 and 1. As a result of distortion, errors are sometimes made during transmission. Consider the following events:

- $U_0$: a 0 is transmitted
- $U_1$: a 1 is transmitted
- $V_0$: a 0 is received
- $V_1$: a 1 is received

The following probabilities are given:
- $P(U_0) = 0.5$, $P(V_0|U_0) = 0.75$, $P(V_0|U_1) = 0.5$

(a) How much information do you gain about which symbol has been received, while you know that a 0 has been transmitted?
(b) How much information do you gain about which symbol has been received, while you know which symbol has been transmitted?
(c) Determine the amount of information that you receive when someone tells you which symbol has been transmitted and which symbol has been received.
(d) Determine the amount of information that you receive when someone tells you which symbol has been transmitted, while you know which symbol has been received.

3. Quadratic and Linear Bounds of Entropy: Prove that:

(a) $H'(p) = \log (1 - p) - \log p$
(b) $H''(p) = -\frac{\log e}{p(1-p)}$
(c) $H(p) \geq 2 \min(p, 1-p)$
(d) $H(p) \geq 1 - 4(p - 1/2)^2$
(e) $H(p) \leq 1 - (2 \log e)(p - 1/2)^2$

4. Search Algorithm: Search algorithms are fundamental in many scientific disciplines. Roughly speaking, a search algorithm can be reduced to a strategy of asking a series of Yes/No questions, then receive corresponding YES/NO answers until the desired item is found. A better search algorithm requires on average a smaller number of questions to find the desired item. For example, let us take the well-known binary search. At every step, we ask the question whether the desired item is in the left branch of the tree, and receive an answer YES or NO. If NO, the we divide the right branch into half, then repeat the procedure again. If YES, we divide the left branch in half and repeat the procedure. We can generalize this concept to the following problem. You suppose you are told that your desired item are in exactly one of the six slots numbered 1, 2, . . . 6. Furthermore, you are told that the probability that your item is in slots 1, 2, 3, 4, 5, 6 are 0.3, 0.2, 0.05, 0.05, 0.25, 0.15, respectively. You want to gain as much information about the location of your item from getting an answer to your first YES/NO question.
(a) What is the first question would you ask? Remember that you must phrase you question such that it can be answered in a form of YES or NO. For example, a valid question would be ”is my item in slot 3 or slot 4 ?” For this example, is there more than one question that you can ask which gives you the same most information? Rigorously (mathematically) justify your approach with information theory.

(b) Re-examine the binary search example in which the question always asks whether the desired item is in the left or right branch, each consists roughly equal number of items. Discuss the assumption on the probability of the desired item in each location.

5. Chapter 2, problem 17

6. Chapter 2, problem 29