Exact Inference: Introduction

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- Using a Bayesian network to compute probabilities is called inference
- In general, inference involves queries of the form:

  P(X | **E**=**e**)

  E = The evidence variable(s)

  X = The query variable(s) (Assume a single variable for now)

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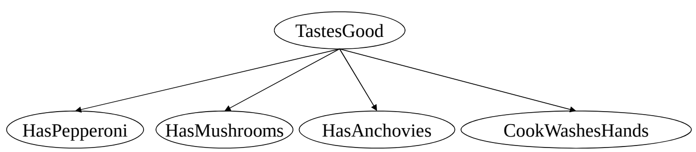

- An example of a query would be:
  P(*TastesGood* = *true* | *HasPepperoni* = *true*, *HasMushrooms* = *true*, *HasAnchovies* = *false*)
- Note: Even though *CookWashesHands* is in the Bayesian network, it is not given values in the query (ie. they do not appear either as query variables or evidence variables)
- They are treated as unobserved variables

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Recall that:

$$\boldsymbol{P}(X \mid \boldsymbol{E}=\boldsymbol{e}) = \alpha \boldsymbol{P}(X, \boldsymbol{E}=\boldsymbol{e})$$
$$= \alpha \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{E}=\boldsymbol{e}, \boldsymbol{Y}=\boldsymbol{y})$$

and $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid parents(X_i))$

Enumeration-Ask algorithm:

Answer queries by computing sums of products of conditional probabilities from the network

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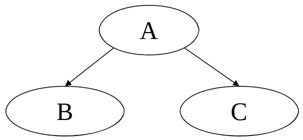

Query: P(A=true | B=true)

How do you solve this? 2 steps:

1. Express it in terms of the joint probability distribution P(A, B, C)
2. Express the joint probability distribution in terms of the entries in the CPTs of the Bayes net

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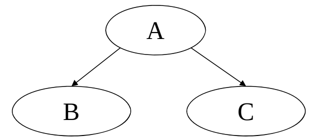

Whenever you see a conditional like P(A=true | B=true), use the Chain Rule:

P(A | B) = P(A, B) / P(B)

$$P(A = true \mid B = true) = \frac{P(A = true, B = true)}{P(B = true)}$$

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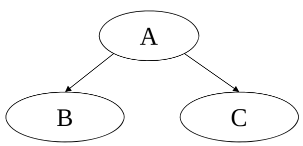

Whenever you need to get a subset of the variables eg. P(B,A) from the full joint distribution P(A,B,C), use marginalization:

$$P(X) = \sum_y P(X, Y = y)$$

$$P(A = true \mid B = true) = \frac{P(A = true, B = true)}{P(B = true)} = \frac{\sum_c P(A = true, B = true, C = c)}{\sum_{a,c} P(A = a, B = true, C = c)}$$

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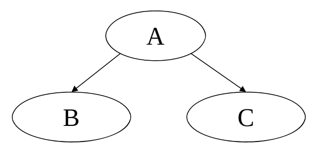

To express the joint probability distribution as the entries in the CPTs, use:

$$P(X_1, \ldots, X_N) = \prod_{i=1}^{N} P(X_i \mid Parents(X_i))$$

$$= \frac{\sum_c P(A = true, B = true, C = c)}{\sum_{a,c} P(A = a, B = true, C = c)}$$

$$= \frac{\sum_c P(C = c \mid A = true) P(B = true \mid A = true) P(A = true)}{\sum_{a,c} P(C = c \mid A = a) P(B = true \mid A = a) P(A = a)}$$

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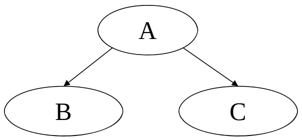

Take the probabilities that don't depend on the terms in the summation and move them outside the summation

$$= \frac{\sum_{c} P(C=c \mid A=true)P(B=true \mid A=true)P(A=true)}{\sum_{a,c} P(C=c \mid A=a)P(B=true \mid A=a)P(A=a)}$$

$$= \frac{P(B=true \mid A=true)P(A=true)\sum_{c} P(C=c \mid A=true)}{\sum_{a} P(B=true \mid A=a)P(A=a)\sum_{c} P(C=c \mid A=a)}$$

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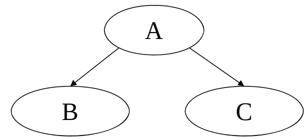

Simplify if possible.

$$= \frac{P(B=true \mid A=true)P(A=true)\sum_{c} P(C=c \mid A=true)}{\sum_{a} P(B=true \mid A=a)P(A=a)\sum_{c} P(C=c \mid A=a)}$$

$$= \frac{P(B=true \mid A=true)P(A=true)}{\sum_{a} P(B=true \mid A=a)P(A=a)}$$

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Exact Inference in graphical models is NP-hard

- Exponential time in worst case

Approximate inference is also NP-hard

- But this is in the worst case. In practice, it is much more efficient

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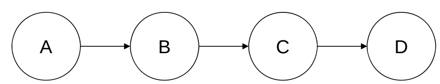
Example #2 (Variable Elimination):

$$P(B) = \sum_{a} P(A=a)P(B \mid A=a)$$



- Note: B is not instantiated with a value. We are computing the table *P(B)*.
- If *A* has *k* values and *B* has *k* values, the number of arithmetic operations required is $O(k^2)$
- If the chain has *n* nodes, computing the joint probability $P(X_1, \ldots, X_n)$ is $O(nk^2)$
- Naïve approach required $O(k^n)$ operations

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$$P(D) = \sum_C \sum_B \sum_A P(A)P(B \mid A)P(C \mid B)P(D \mid C)$$

$$= \sum_C P(D \mid C) \sum_B P(C \mid B) \sum_A P(A)P(B \mid A)$$

Use dynamic programming to work from the innermost summation outward.

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$$P(D) = \sum_C \sum_B \sum_A P(A)P(B \mid A)P(C \mid B)P(D \mid C)$$

$$= \sum_C P(D \mid C) \sum_B P(C \mid B) \sum_A P(A)P(B \mid A)$$

$$= \sum_C P(D \mid C) \sum_B P(C \mid B)\tau_1(B)$$

$$= \sum_C P(D \mid C)\tau_2(C)$$

$$\psi_1(A,B) = P(A)P(B \mid A)$$
$$\tau_1(B) = \sum_A \psi_1(A,B)$$

$$\psi_2(B,C) = \tau_1(B)P(C \mid B)$$
$$\tau_2(C) = \sum_B \psi_2(B,C)$$

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Two key ideas to variable elimination:

1. Due to structure of BN, some subexpressions in the joint only depend on a small number of variables
2. Dynamic programming caches the intermediate results to avoid recomputing them exponentially many times

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