# Exact Inference: Variable Elimination I

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## Variable Elimination

Recall:

- Let ***X*** be a set of random variables
- A factor $\phi$ is a function from Val(***X***) $\rightarrow \mathcal{R}$
- The set of variables ***X*** is called the scope of the factor and denoted *Scope[*$\phi$*]*

We will be manipulating factors

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# Variable Elimination

Let ***X*** be a set of variables, and Y $\notin$ ***X*** a variable. Let $\phi(\boldsymbol{X}, Y)$ be a factor. We define the factor marginalization of *Y* in $\phi$, denoted $\Sigma_Y \phi$ to be a factor $\psi$ over ***X*** such that:

$$\psi(\boldsymbol{X}) = \sum_{Y} \phi(\boldsymbol{X}, Y)$$

This operation is also called summing out of *Y* in $\psi$

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# Variable Elimination

| A | B | C | P(A,B,C) |
|---|---|---|---|
| 1 | 1 | 1 | 0.25 |
| 1 | 1 | 2 | 0.35 |
| 1 | 2 | 1 | 0.08 |
| 1 | 2 | 2 | 0.16 |
| 2 | 1 | 1 | 0.05 |
| 2 | 1 | 2 | 0.07 |
| 2 | 2 | 1 | 0 |
| 2 | 2 | 2 | 0 |
| 3 | 1 | 1 | 0.15 |
| 3 | 1 | 2 | 0.21 |
| 3 | 2 | 1 | 0.09 |
| 3 | 2 | 2 | 0.18 |

Summing out B

| A | C | P(A,C) |
|---|---|---|
| 1 | 1 | 0.25+0.08=0.33 |
| 1 | 2 | 0.35+0.16=0.51 |
| 2 | 1 | 0.05+0=0.05 |
| 2 | 2 | 0.07+0=0.07 |
| 3 | 1 | 0.15+0.09=0.24 |
| 3 | 2 | 0.21+0.18=0.39 |

Note: we only sum up entries in the table where the values of **X** match up.

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# Variable Elimination

In a Bayesian network:

- Summing out all variables results in a factor with value 1

In a Markov network:

- Summing out all variables in the unnormalized distribution $\tilde{P}_\Phi$ defined by the product of factors in the Markov network results in the partition function

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# Variable Elimination

Recall: Let ***X***, ***Y***, and ***Z*** be three disjoint sets of variables, and let $\phi_1(\mathbf{X},\mathbf{Y})$ and $\phi_2(\mathbf{Y},\mathbf{Z})$ be two factors. We define the factor product $\phi_1 \times \phi_2$ to be a factor $\psi$: $Val(\mathbf{X},\mathbf{Y},\mathbf{Z}) \rightarrow \mathcal{R}$ as follows:

$$\psi(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \phi_1(\mathbf{X},\mathbf{Y})\ \phi_2(\mathbf{Y},\mathbf{Z})$$

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# Variable Elimination

Example of a factor product:

| A | B | $\phi_1$(A,B) |
|---|---|---|
| 0 | 0 | 0.5 |
| 0 | 1 | 0.8 |
| 1 | 0 | 0.1 |
| 1 | 1 | 0 |
| 2 | 0 | 0.3 |
| 2 | 1 | 0.9 |

| B | C | $\phi_2$(B,C) |
|---|---|---|
| 0 | 0 | 0.5 |
| 0 | 1 | 0.7 |
| 1 | 0 | 0.1 |
| 1 | 1 | 0.2 |

| A | B | C | $\psi$(A,B,C) |
|---|---|---|---|
| 0 | 0 | 0 | (0.5)(0.5)=0.25 |
| 0 | 0 | 1 | (0.5)(0.7)=0.35 |
| 0 | 1 | 0 | (0.8)(0.1)=0.08 |
| 0 | 1 | 1 | (0.8)(0.2)=0.16 |
| 1 | 0 | 0 | (0.1)(0.5)=0.05 |
| 1 | 0 | 1 | (0.1)(0.7)=0.07 |
| 1 | 1 | 0 | (0)(0.1)=0 |
| 1 | 1 | 1 | (0)(0.2)=0 |
| 2 | 0 | 0 | (0.3)(0.5)=0.15 |
| 2 | 0 | 1 | (0.3)(0.7)=0.21 |
| 2 | 1 | 0 | (0.9)(0.1)=0.09 |
| 2 | 1 | 1 | (0.9)(0.2)=0.18 |

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# Variable Elimination

Operations over factors:

- Addition is commutative: $\sum_X \sum_Y \phi = \sum_Y \sum_X \phi$
- Multiplication is commutative: $\phi_1 \cdot \phi_2 = \phi_2 \cdot \phi_1$
- Products are associative: $(\phi_1 \cdot \phi_2) \cdot \phi_3 = \phi_1 \cdot (\phi_2 \cdot \phi_3)$
- Exchanging summations and products:

  If $X \neq Scope[\phi_1]$, $\sum_X (\phi_1 \cdot \phi_2) = \phi_1 \cdot \sum_X \phi_2$

  ( X is not in the terms of $\phi_1$)

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# Variable Elimination

Example: $$P(D) = \sum_C \sum_B \sum_A P(A, B, C, D)$$

$$= \sum_C \sum_B \sum_A \varphi_A \cdot \varphi_B \cdot \varphi_C \cdot \varphi_D$$

$$= \sum_C \sum_B \varphi_C \cdot \varphi_D \cdot \varphi_B \left( \sum_A \varphi_A \right)$$

$$= \varphi_D \sum_C \varphi_C \cdot \left( \sum_B \varphi_B \cdot \left( \sum_A \varphi_A \right) \right)$$

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# Variable Elimination

The general problem involves a sum-product inference task:

$$\sum_{\mathbf{Z}} \prod_{\phi \in \Phi} \phi$$

Trick: Push in the summations as far as you can

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# Variable Elimination

**Procedure** Sum-Product-VE(
    $\Phi$,    // A set of factors
    ***Z***,    // Set of variables to be eliminated
    <    // Ordering on **Z**
)

1. Let $Z_1, \ldots, Z_k$ be an ordering of **Z** such that
2. $Z_i < Z_j$ if and only if $i < j$
3. **for** i = 1, … k
4. $\Phi \leftarrow$ Sum-Product-Eliminate-Var($\Phi$,$Z_i$)
5. $\phi^* \leftarrow \prod_{\phi \in \Phi} \phi$
6. Return $\phi^*$



# Variable Elimination

**Procedure** Sum-Product-Eliminate-Var(
    $\Phi$,    // A set of factors
    Z,    // Variable to be eliminated
)

1. $\Phi' \leftarrow \{\phi \in \Phi : Z \in Scope[\phi]\}$
2. $\Phi'' \leftarrow \Phi - \Phi'$
3. $\psi \leftarrow \prod_{\phi \in \Phi'} \phi$
4. $\tau \leftarrow \sum_{Z} \psi$
5. Return $\Phi'' \cup \{\tau\}$

## Variable Elimination

Let ***X*** be some set of variables, and let $\Phi$ be a set of factors such that for each $\phi \in \Phi$, *Scope[*$\Phi$*]* $\subseteq$ ***X***. Let ***Y*** $\subset$ ***X*** be a set of query variables, and let ***Z*** = ***X*** – ***Y***. Then for any ordering < over ***Z***, *Sum-Product-VE*($\Phi$,***Z***,<) returns a factor $\phi^*$(***Y***) such that

$$\phi^*(\mathbf{Y}) = \sum_{\mathbf{Z}} \prod_{\phi \in \Phi} \phi$$

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## Variable Elimination

Example: compute $P_{\mathcal{B}}(Y)$ for Bayesian network $\mathcal{B}$. Let:

- $\Phi = \{\phi_{X_i}\}_{i=1}^{n}$ where $\phi_{X_1} = P(X_i \mid Parents(X_i))$
- ***Z***=$\{Z_1,\ldots,Z_m\}$ = $\mathcal{X}$- ***Y*** (eliminate all non-query variables)

Note: We can do the exact same thing on a Markov network except the final factor $\phi^*$*(**Y**)* is unnormalized.

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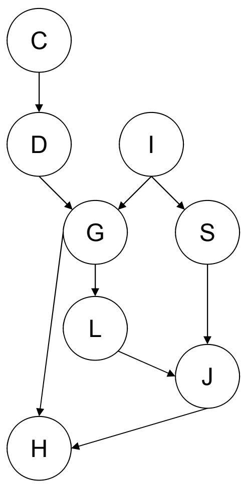
# Variable Elimination




We will compute *P(J)* using the elimination ordering *C*, *D*, *I*, *H*, *G*, *S*, *L*.

Note that:

P(C,D,I,G,S,L,J,H)

= P(C) P(D|C) P(I) P(G|I,D) P(S|I) P(L|G) P(J|L,S) P(H|G,J)

= $\phi_C(C)\ \phi_D(D,C)\ \phi_I(I)\ \phi_G(G,I,D)\ \phi_S(S,I)\ \phi_L(L,G)\ \phi_J(J.L,S)\ \phi_H(H,G,J)$

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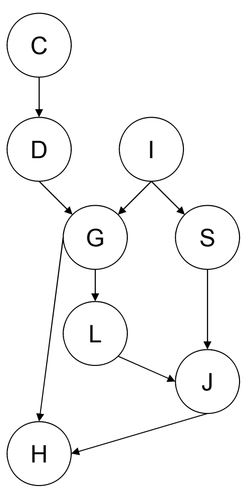
# Variable Elimination




Elimination ordering: *C*, *D*, *I*, *H*, *G*, *S*, *L*

1. Eliminating C:

$$\Psi_1(C,D) = \phi_C(C)\cdot\phi_D(D,C)$$

$$\tau_1(D) = \sum_C \Psi_1(C,D)$$

2. Eliminating D:

$$\Psi_2(G,I,D) = \phi_G(G,I,D)\cdot\tau_1(D)$$

$$\tau_2(G,I) = \sum_D \Psi_2(G,I,D)$$

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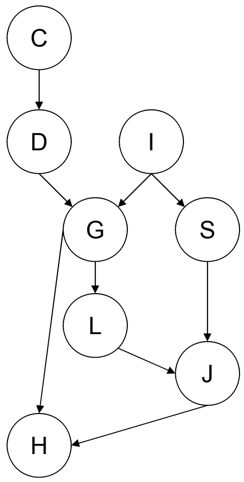
# Variable Elimination




Elimination ordering: *C*, *D*, *I*, *H*, *G*, *S*, *L*

3. Eliminating I:

$$\Psi_3(G,I,S) = \phi_I(I) \cdot \phi_S(S,I) \cdot \tau_2(G,I)$$

$$\tau_3(G,S) = \sum_I \Psi_3(G,I,S)$$

4. Eliminating H:

$$\Psi_4(G,J,H) = \phi_H(H,G,J)$$

$$\tau_4(G,J) = \sum_H \Psi_4(G,J,H)$$

Note: $\tau_4 \equiv 1$ since $\sum_H P(H|G,J)$. However, in this elimination ordering, you do need to generate this factor for the next step.

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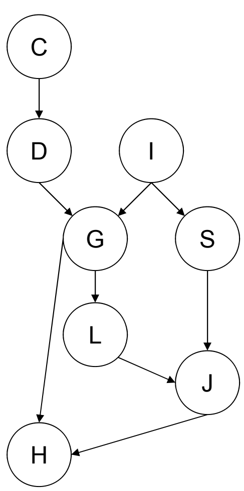
# Variable Elimination




Elimination ordering: *C*, *D*, *I*, *H*, *G*, *S*, *L*

5. Eliminating G:

$$\Psi_5(G,J,L,S) = \tau_4(G,J) \cdot \tau_3(G,S) \cdot \phi_L(L,G)$$

$$\tau_5(J,L,S) = \sum_G \Psi_5(G,J,L,S)$$

6. Eliminating S:

$$\Psi_6(J,L,S) = \tau_5(J,L,S) \cdot \phi_J(J,L,S)$$

$$\tau_6(J,L) = \sum_S \Psi_6(J,L,S)$$

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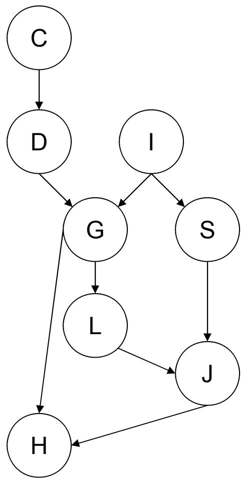
# Variable Elimination




Elimination ordering: *C*, *D*, *I*, *H*, *G*, *S*, *L*

7. Eliminating L:

$\Psi_7(J, L) = \tau_6(J, L)$

$\tau_7(J) = \sum_L \Psi_7(J, L)$

Note: You can use any elimination ordering eg. G, I, S, L, H, C, D. This is a bad ordering because it produces factors with very large scope (see Table 9.2 pg 302)

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# Variable Elimination

How do we deal with evidence?

eg. P( J = true | I = high, H = false)?

Note that:

$$P(J \mid I = high, H = false) = \frac{P(J, I = high, H = false)}{P(I = high, H = false)}$$

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## Variable Elimination

Proposition 4.7:

Let $\mathcal{B}$ be a Bayesian network over $\mathcal{X}$ and ***E*** = ***e*** an observation. Let ***W*** = $\mathcal{X}$ – ***E***. Then $P_{\mathcal{B}}$(***W*** |**e**) is a Gibbs distribution defined by the factors where

$$\phi_{X_i} = P_{\mathcal{B}}(X_i \mid Parents(X_i))[\boldsymbol{E} = \boldsymbol{e}]$$

This is a reduced factor, meaning it is the factor with entries inconsistent with E=e removed.

The partition function for this Gibbs distribution is *P*(***e***)

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## Variable Elimination

- Replace each factor $\phi$ with $\phi$[**E=e**]
- Then do variable elimination like normal
- Remember to normalize with $P(E = e)$

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# Variable Elimination

Procedure Cond-Prob-VE(

$\mathcal{K}$, // A network over $\mathcal{X}$

***Y***, // Set of query variables

***E*** =***e*** // Evidence

)

*1.* $\Phi \leftarrow$ Factors parameterizing K
2. Replace each $\phi \in \Phi$ by $\phi$ [***E***=***e***]
3. Select an elimination ordering <
***4.*** $\boldsymbol{Z} \leftarrow \mathcal{X} - \boldsymbol{Y} - \boldsymbol{E}$
*5.* $\phi^* \leftarrow$Sum-Product-VE($\Phi$,<,***Z***)
*6.* $\alpha \leftarrow \sum_{y \in Val(\boldsymbol{Y})} \phi^* (\boldsymbol{y})$
7. return $\alpha$, $\phi^*$

Note: $\phi^*$ represents *P(**Y**,**e**)* so divide $\phi^*$ by $\alpha$ to get *P(**Y**|**e**)*

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# Variable Elimination

Computing: P(J,I=high,H=false)

| Step | Variable Eliminated | Factors Used | Variables Involved | New Factor |
|---|---|---|---|---|
| 1' | C | $\phi_C$(C), $\phi_D$(D,C) | C, D | $\tau_1$'(D) |
| 2' | D | $\phi_G$[I=high](G,D), $\phi_I$[I=high](), $\tau_1$'(D) | G, D | $\tau_2$'(G) |
| 5' | G | $\tau_2$'(G), $\phi_L$(L,G), $\phi_H$[H=false](G.J) | G, L, J | $\tau_5$'(J,L) |
| 6' | S | $\phi_S$[I=high](S), $\phi_J$(J, L, S) | J, L, S | $\tau_6$'(J,L) |
| 7' | L | $\tau_6$'(J,L), $\tau_5$'(J,L) | J,L | $\tau_7$'(J) |

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