Undirected Graphical Models 1

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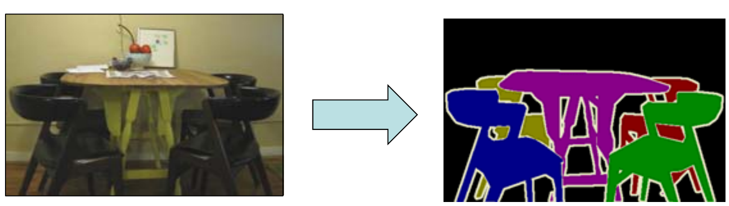
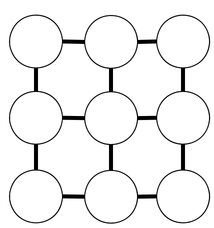
Symmetric interactions (Examples)

Image Segmentation (From PASCAL VOC 2011 data)



Each node in this undirected graphical model is a pixel / region



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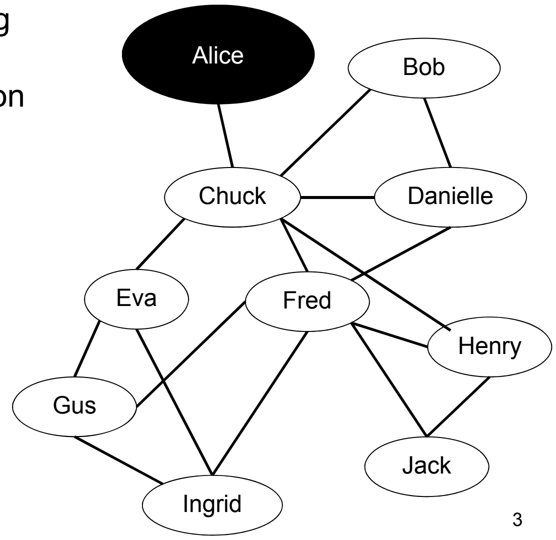
Symmetric interactions (Examples)

Social network modeling
- Marketing
- Insider threat detection
- Fraud detection



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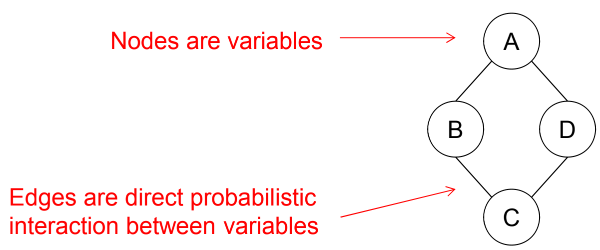
Introduction

Markov network



What about the parameters?
- Standard CPD doesn't work – no notion of a "parent"
- Need a more symmetric parameterization

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Introduction

Let $\boldsymbol{D}$ be a set of random variables. We define a factor ϕ to be a function from $Val(\boldsymbol{D}) \rightarrow \Re$. A factor is nonnegative if all its entries are nonnegative.

The set of variables $\boldsymbol{D}$ is called the scope of the factor and denoted $Scope[\phi]$

Unless stated otherwise, we restrict attention to nonnegative factors

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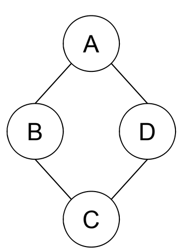
Introduction



A	B	$\phi_1(A,B)$
0	0	30
0	1	5
1	0	1
1	1	10

B	C	$\phi_2(B,C)$
0	0	100
0	1	1
1	0	1
1	1	100

C	D	$\phi_3(C,D)$
0	0	1
0	1	100
1	0	100
1	1	1

D	A	$\phi_4(D,A)$
0	0	100
0	1	1
1	0	1
1	1	100

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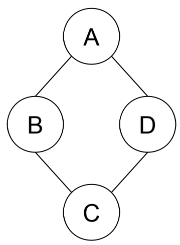
Introduction

Think of $\phi_1(A, B)$ like an unnormalized joint distribution between A and B. This column doesn't have to sum to 1



A	B	$\phi_1(A, B)$
0	0	30
0	1	5
1	0	1
1	1	10

The bigger the value, the more likely the configuration eg. A = 0, B = 0 is the most likely

I can increase this value to make A=1 and B=1 more likely but it is not clear how this affects the full joint distribution between A, B, C, and D

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Introduction

Because the factors are not normalized, need to normalize everything at the end to produce a probability distribution.

$$P(a, b, c, d) = \frac{1}{Z} \emptyset_1(a, b)\, \emptyset_2(b, c) \emptyset_3(c, d) \emptyset_4(d, a)$$

$$Z = \sum_{a,b,c,d} \emptyset_1(a, b)\, \emptyset_2(b, c) \emptyset_3(c, d) \emptyset_4(d, a)$$

Normalizing constant (also called the partition function). Can be difficult to compute!

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Introduction

Connections between factorization and independence properties

- Structure of the factors allows us to decompose the distribution
- $P \models (\boldsymbol{X} \perp \boldsymbol{Y} | \boldsymbol{Z})$ iff $P(\mathcal{X}) = \phi_1(\boldsymbol{X}, \boldsymbol{Z})\phi_2(\boldsymbol{Y}, \boldsymbol{Z})$ Independence properties of the distribution *P* correspond to separation properties of the graph *G* over which *P* factorizes

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Parameterizations

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Parameterization

- Factors subsume (generalize) the notion of a joint distribution:
  - A joint distribution over $\boldsymbol{D}$ is a factor over $\boldsymbol{D}$
- Factors subsume a conditional probability distribution (CPD)
  - A CPD $P(X|\boldsymbol{U})$ is a factor over $\{X\} \cup \boldsymbol{U}$.
  - A CPD is a special case of a factor that is normalized

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Parameterization

Let $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ be three disjoint sets of variables, and let $\phi_1(\boldsymbol{X}, \boldsymbol{Y})$ and $\phi_2(\boldsymbol{Y}, \boldsymbol{Z})$ be two factors. We define the factor product $\phi_1 \times \phi_2$ to be a factor Ψ: $Val(X, Y, Z) \rightarrow \Re$ as follows:

$$\Psi(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = \phi_1(\boldsymbol{X}, \boldsymbol{Y})\phi_2(\boldsymbol{Y}, \boldsymbol{Z})$$

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Parameterization

Example of a factor product:

A	B	$\phi_1(A,B)$
0	0	0.5
0	1	0.8
1	0	0.1
1	1	0
2	0	0.3
2	1	0.9

B	C	$\phi_2(B,C)$
0	0	0.5
0	1	0.7
1	0	0.1
1	1	0.2

A	B	C	$\Psi(X,Y,Z)$
0	0	0	(0.5)(0.5)=0.25
0	0	1	(0.5)(0.7)=0.35
0	1	0	(0.8)(0.1)=0.08
0	1	1	(0.8)(0.2)=0.16
1	0	0	(0.1)(0.5)=0.05
1	0	1	(0.1)(0.7)=0.07
1	1	0	(0)(0.1)=0
1	1	1	(0)(0.2)=0
2	0	0	(0.3)(0.5)=0.15
2	0	1	(0.3)(0.7)=0.21
2	1	0	(0.9)(0.1)=0.09
2	1	1	(0.9)(0.2)=0.18

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Parameterizations

For Bayesian Networks;

- Since CPDs and joint distributions are factors
- Chain rule for BNs can be thought of as the product of CPD factors
- Letting $\phi_{X_i}(X_i, Parents(X_i))$ =$P(X_i|Parents(X_i))$

$$P(X_1, \dots, X_N) = \prod_i \phi_{X_i}(X_i, Parents(X_i))$$

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Parameterizations

A distribution P_Φ is a Gibbs distribution parameterized by a set of factors $\Phi = \{\phi_1(\boldsymbol{D_1}), \ldots, \phi_K(\boldsymbol{D_K})\}$ if it is defined as follows:

$$P_\Phi(X_1, \ldots, X_n) = \frac{1}{Z} \tilde{P}_\Phi(X_1, \ldots, X_n)$$

where

$$\tilde{P}_\Phi(X_1, \ldots, X_n) = \phi_1(\boldsymbol{D_1}) \times \phi_2(\boldsymbol{D_2}) \times \ldots \times \phi_K(\boldsymbol{D_K})$$

is an unnormalized measure and

$$Z = \sum_{X_1, \ldots, X_n} \tilde{P}_\Phi(X_1, \ldots, X_n)$$

is a normalizing constant called the partition function

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Parameterizations

We say that a distribution P_Φ with $\Phi = \{\phi_1(\boldsymbol{D_1}), \ldots, \phi_K(\boldsymbol{D_K})\}$ factorizes over a Markov network $\mathcal{H}$ if each $\boldsymbol{D_k}$ (k=1, …, K) is a complete subgraph (or clique) of $\mathcal{H}$

A complete subgraph (or clique) is a fully connected subgraph

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Parameterizations

The terms that you multiply together for the joint distribution of a Markov network are often called clique potentials

$$P(X_1, \ldots, X_N) = \frac{1}{Z}\phi_1(\boldsymbol{C_1}) \times \phi_2(\boldsymbol{C_2}) \times \ldots \times \phi_K(\boldsymbol{C_K})$$

Clique Potential

Confusing point: A clique potential can be made up of a product of factors. Suppose clique C_1 has scope A, B and C. The clique potential for C_1 could be $\phi_1(A,B) \times \phi_2(B,C) \times \phi_3(A,C)$.

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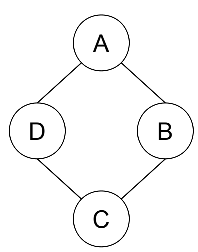
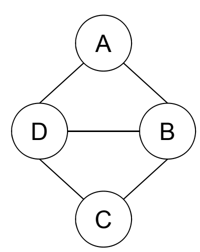
Parameterizations

Examples of Markov networks and their cliques





<u>Cliques:</u>

{A,B}, {B,C}, {C,D}, {A,D}

<u>Cliques:</u>

{A,B,D}, {B,C,D},

{A,D},{C,D},{A,B},{B,C}{B,D}

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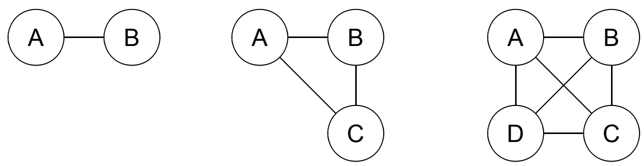
Parameterizations

Note: every complete subgraph is a subset of some (maximal) clique eg.



Because of this, we can reduce the number of factors in our parameterization by allowing factors only for maximal cliques

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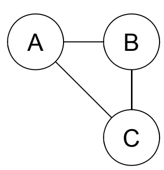
Parameterizations



The maximal clique for this graph has scope A, B, C.

You can parameterize this in two ways:

1. $P_{\Phi}(A, B, C) = \phi_1(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$

or

2. $P_{\Phi}(A, B, C) = \phi_1(\boldsymbol{A}, \boldsymbol{B}) \times \phi_2(\boldsymbol{B}, \boldsymbol{C}) \times \phi_3(\boldsymbol{A}, \boldsymbol{C})$

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Finer-Grained Parameterization

- Markov network structure does not reveal whether the factors in the parameterization involve maximal cliques or subsets of these cliques
- Factor graph makes this explicit in the structure.

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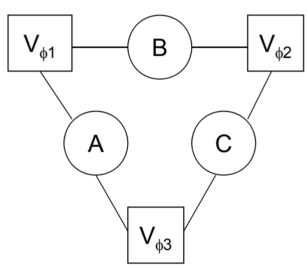
Finer-Grained Parameterizations

A factor graph *F* is an undirected graph containing two types of nodes:

- Variable nodes (denoted as ovals) and
- Factor nodes (denoted as squares).

The graph only contains edges between variable nodes and factor nodes.



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Finer-Grained Parameterizations

A factor graph F is parameterized by a set of factors, where each factor node V_ϕ is associated with only one factor ϕ, whose scope is the set of variables that are neighbors of V_ϕ in the graph.

A distribution P factorizes over F if it can be represented as a set of factors of this form.

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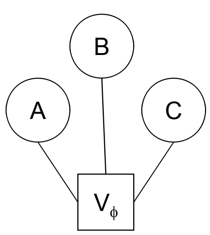
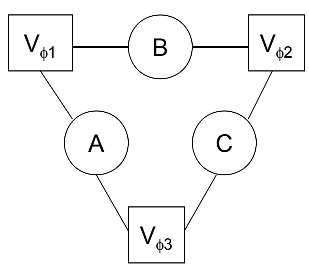
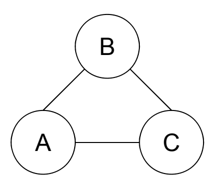
Finer-grained Parameterization



A single factor over all three variables



3 pairwise factors



The induced Markov network

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Finer-grained Parameterizations

Rather than encoding factors as complete tables over the scope of the factor, we can use a log-linear model:

$\phi(\boldsymbol{D}) = \exp(-\varepsilon(\boldsymbol{D}))$

Where $\varepsilon(\boldsymbol{D}) = -\ln \phi(\boldsymbol{D})$ is an energy function (which you want to minimize)

$$P(X_1, \ldots, X_n) \propto \exp\left[-\sum_{i=1}^{m} \varepsilon_i(\boldsymbol{D}_i)\right]$$

Noote: log representation makes sure the distribution is positive

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Finer-grained Parameterizations

Let ***D*** be a subset of variables. We define a feature f(***D***) to be a function from $\boldsymbol{D} \rightarrow R$.

eg. an indicator feature takes on value 1 for some values $\boldsymbol{y} \in \text{Val}(\boldsymbol{D})$ and 0 otherwise

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Finer-grained Parameterizations

Features provide a compact way to specify certain types of interactions

Example: Suppose A_1 and A_2 can take on l possible values a^1, .., a^l . A_1 and A_2 prefer situations when they take on the same value, and have no preference otherwise. The energy function might take the following:

$$\varepsilon(A_1, A_2) = \begin{cases} -10 & A_1 = A_2 \\ 0 & \text{otherwise} \end{cases}$$

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Finer-grained Parameterizations

(example continued)

Two options for representing the factor:

- As a table, it requires l^2 values
- Log-linear function in terms of a feature $f(A_1,A_2)$ that is an indicator function for the event A_1=A_2. The energy function looks like:

$$\varepsilon(A_1, A_2) = 3 * I(A_1 = A_2)$$

We just replaced a table with a function

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Finer-grained Parameterizations

A distribution P is a log-linear model over a Markov network $\mathcal{H}$ if it is associated with:

- A set of features $F = \{f_1(\mathbf{D}_1), \ldots, f_k(\mathbf{D}_k)\}$, where each $\boldsymbol{D}_i$ is a complete subgraph in $\mathcal{H}$
- A set of weights $w_1, \ldots, w_k$

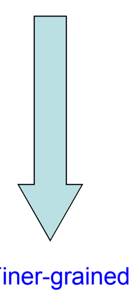
Such that

$$P(X_1, \ldots, X_n) = \frac{1}{Z} \exp\left[-\sum_{i=1}^{k} w_i f_i(\boldsymbol{D}_i) \right]$$

Finer-grained Parameterizations

3 representations of the parameterization of a Markov network:

1. Markov network: product over potentials on cliques
2. Factor graph: product of factors
3. Set of features: product over feature weights



Which is most appropriate? Depends on the nature of the problem…