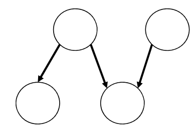
Undirected Graphical Models 3
Bayesian Networks and Markov Networks

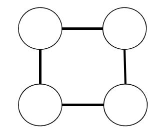
Bayesian Networks and Markov Networks

We have now seen two types of graphical models for representing joint probability distributions:

Bayesian Networks



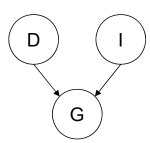
Markov Networks



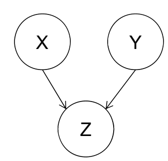
Can we convert from one type to the other?

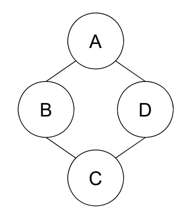
Bayesian Networks and Markov Networks

- Can a Markov network represent any Bayesian network? No!
- Can't use a Markov network to represent a distribution corresponding a Bayesian network v-structure



Bayesian Networks and Markov Networks

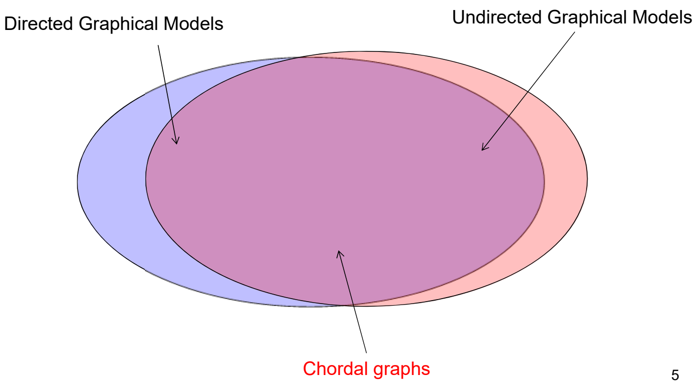




Bayesian networks can represent independence constraints that Markov networks cannot

Markov networks can represent independence constraints that Bayesian networks cannot

Bayesian Networks and Markov Networks



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Bayesian Networks to Markov Networks

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BNs to MNs

To convert a Bayesian Network B to a Markov Network H, we can think of B as a Gibbs distribution with:

- Each factor ϕ_{X_i} in H corresponds to each conditional probability table $P(X_i|Parents(X_i))$ in B
- The scope of factor ϕ_{X_i} is $X_i \cup Parents(X_i)$
- The partition function $Z = 1$

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BNs to MNs

How do we create an undirected graph that is an I-map for the distribution P_B represented by the BN?

Need to moralize each factor:

- Add an edge between X_i and its parents
- Add an edge between all the parents of X_i

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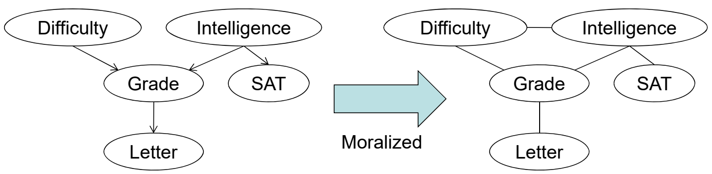
BNs to MNs

The moral graph $M[G]$ of a Bayesian network structure G over $\boldsymbol{X}$ is the undirected graph over $\boldsymbol{X}$ that contains an undirected edge between X and Y if:

- There is a directed edge between them in G (in either direction)
- Or X and Y are both parents of the same node



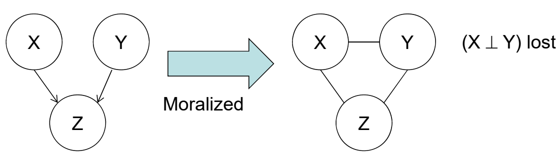
BNs to MNs

Let G be any Bayesian network graph. The moralized graph $M[G]$ is a minimal I-map for G.

[See proof of Proposition 4.8 in textbook]

BNs to MNs

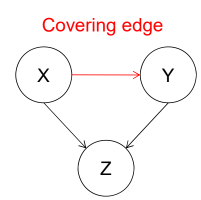
- The addition of the moralizing edges to the MN H leads to the loss of independence information implied by the graph structure



- Only happens if moralization adds new edges into the graph

BNs to MNs

A Bayesian network G is moral if it contains no immoralities ie. for any pair of variables X, Y that share a child, there is a covering edge between X and Y



BNs to MNs

- If the directed graph G is moral, then its moralized graph $M[G]$ is a perfect map of G.
- Unfortunately, very few directed graphs are moral

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Markov Networks to Bayesian Networks

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MNs to BNs

- Much harder (conceptually and computationally) to find a Bayesian network that is a minimal I-map for a Markov network
- Resulting Bayesian network might have (many) more edges than the Markov network

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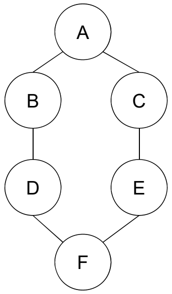
MNs to BNs

Example: Find a Bayesian network I-map for the given Markov Network



Build-Minimal-I-Map (from Section 3.4.1):
To find a minimal I-map for a distribution *P*:
- Pick a variable ordering
- For each variable X_i in the ordering:
  - Find some minimal subset $\boldsymbol{U}$ of $\{X_1, \dots, X_{i-1}\}$ to be X_i's parents in G such that $\{X_i \perp \{X_1, \dots, X_{i-1}\} - \boldsymbol{U} | \boldsymbol{U}\}$

Note: $(X_i \perp \emptyset | \{X_1, \dots X_{i-1}\})$ is trivially true

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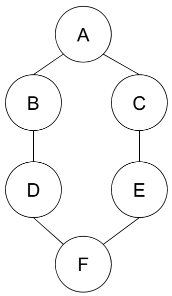
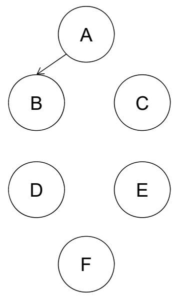
MNs to BNs

Example: Find a Bayesian network I-map for the given Markov Network

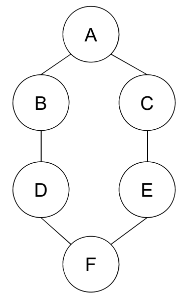


Ordering: A,B,C,D,E,F



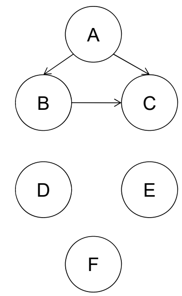
MNs to BNs

Example: Find a Bayesian network I-map for the given Markov Network



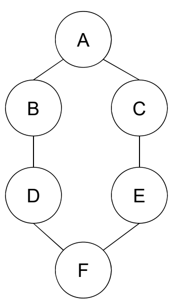
Ordering: A,B,C,D,E,F

(C not ⊥ B | A), so must add B as parent of C



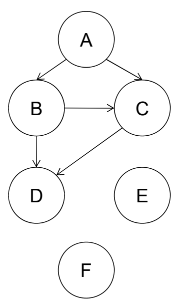
MNs to BNs

Example: Find a Bayesian network I-map for the given Markov Network



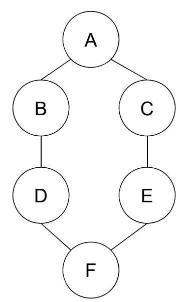
Ordering: A,B,C,D,E,F

(D not ⊥ C | B), so must Add C as parent of B. (D ⊥ A | {B,C})



MNs to BNs

Example: Find a Bayesian network I-map for the given Markov Network

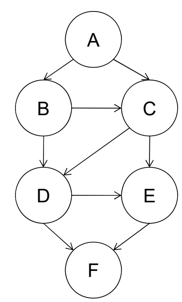


Ordering: A,B,C,D,E,F

And so on…

Notice the edges added which result in a set of triangles.

The graph is chordal (defn to follow)



MNs to BNs

- Let $X_1—X_2—X_3—X_4$ be a loop in a graph
- A chord in the loop is an edge connecting X_i and X_j for two nonconsecutive nodes X_i, X_j.
- An undirected graph H is said to be chordal if any loop $X_1—X_2—\ldots—X_k—X_1$ for $K \geq 4$ has a chord

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MNs to BNs

Theorem 4.10: Let H be a Markov network structure, and let G be any Bayesian network minimal I-map for H. Then G can have no immoralities. [Proof omitted]

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MNs to BNs

Any nontriangulated loop of length at least 4 in a Bayesian network graph necessarily contains an immorality.

Corollary 4.3: Let H be a Markov network structure, and let G be any minimal I-map for H. Then G is necessarily chordal.

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MNs to BNs

- Turning a MN to a BN requires triangulation: adding enough edges to a graph to make it chordal.
- Leads to the loss of independence information
- When converting from BN→MN, the (moralizing) edges added are in some sense implicitly there (ie. each factor in the BN involves a node and its parents)
- When converting from MN→BN, we can introduce a large number of edges (via triangulation) which results in very large cliques

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Chordal Graphs

Chordal Graphs

Let H be a chordal Markov network. Then there is a Bayesian network G which is a perfect map for H ie. $I(H) = I(G)$

We will introduce concepts to help us sketch out a proof for this.

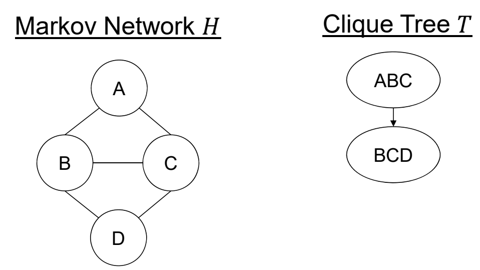
Chordal Graphs

Basic idea:

- Show that any connected chordal Markov network H can be decomposed into a clique tree (to be defined)
- The clique tree encodes independencies in H
- These independencies can be represented in a Bayesian network G

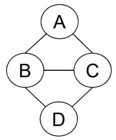
Chordal Graphs

- Let H be a connected undirected graph
- Let $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_k$ be the set of maximal cliques in H
- Let T be any tree-structured graph whose nodes correspond to the maximal cliques $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_k$

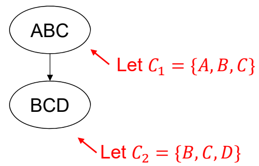


Chordal Graphs

Markov Network H



Clique Tree T



- Define the Sepset $S_{1,2} = C_1 \cap C_2$ (Note: the two cliques must be connected by an edge)
- Let $\boldsymbol{W}_{<(1,2)} = \{A\}$ be all of the variables that appear in any clique on the $\boldsymbol{C}_1$ side of the edge
- Let $\boldsymbol{W}_{<(2,1)} = \{D\}$ be all of the variables that appear in any clique on the $\boldsymbol{C}_2$ side of the edge

Chordal Graphs

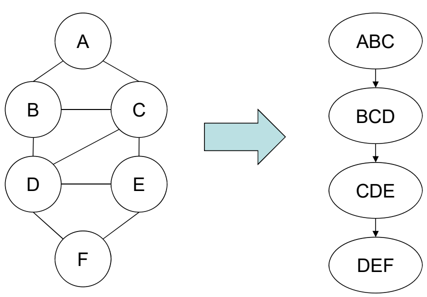
We say that a tree T is a clique tree for H if:

- Each node corresponds to a clique in H, and each maximal clique in H is a node in T
- Each sepset $S_{i,j}$ separates $W_{<(i,j)}$ and $W_{<(j,i)}$ in H
- Note: $W_{<(i,j)} \perp W_{<(j,i)} | S_{i,j}$

Theorem 4.12: Every undirected chordal graph H has a clique tree T.
[See book for proof by induction]

Chordal Graphs

How do you form a clique tree from bigger Markov networks?



Pick a clique as the root. Then pick an ordering of the cliques that go out from the root (called a topological ordering)

Chordal Graphs

Theorem 4.13: Let H be a chordal Markov network. Then there is a Bayesian network G such that $I(H) = I(G)$.

The process [see full proof in book]:

- Build clique tree T from H
- Call Build-Minimal-I-Map to get Bayesian network G
- G (moralized Baysian network) has the same edges as H due to sepset property $W_{<(i,j)} \perp W_{<(j,i)} | S_{i,j}$