Kalman Filter

\[ X_{n+1} = AX_n + WN \]
\[ Y_n = Cx_n + VN \]

\[ Z = Y_n \]

Goal: Compute \( P(X_{n+1} | X_n) \)
Denote \( X_{n|m} \) as \( X_n \) for \( m \) between \( n \) and \( n+1 \)

(1) Kalman Filtering Algorithm:
- Compute Prediction \( P(X_{n+1} | X_n) \)
- Compute Correction \( \hat{P}(X_{n+1} | X_n) \) based on \( \hat{P}(X_{n+1}) \)

From
\[ (u_0, y_0, x_0) \quad \ldots \quad (u_n, y_n, x_n) \]

Repeat(1)

(2) and (3)
From \( P(X_{n+1}) \) compute \( X_{n+1|n} \):

\[
P(X_{n+1|n}) = N \left( E[X_{n+1}] + W_n \right)
\]

\[
E[X_{n+1}] = A \hat{X}_n + W_n 
\]

\[
\hat{X}_n = A \hat{X}_{n-1} + V_n 
\]

Correction:

\[
Y_{n+1|n} = N \left( E[CX_{n+1}] + C \hat{P}_{n+1|n} + V_{n+1} \right)
\]

\[
C \hat{X}_{n+1} = C \hat{X}_n + V_{n+1} 
\]
Look at the joint distribution of $X_{n+1|n}$ and $Y_{n+1|n}$:

$$(X_{n+1|n}, Y_{n+1|n}) = N\left(\begin{pmatrix} E[X_{n+1|n}] \\ E[Y_{n+1|n}] \end{pmatrix}, \begin{pmatrix} \text{Var}(X_{n+1|n}) & \text{Cov}(X_{n+1|n}, Y_{n+1|n}) \\ \text{Cov}(X_{n+1|n}, Y_{n+1|n}) & \text{Var}(Y_{n+1|n}) \end{pmatrix}\right)$$

$$= N\left(\begin{pmatrix} \hat{X}_{n+1|n} \\ \hat{Y}_{n+1|n} \end{pmatrix}, \begin{pmatrix} P_{n+1|n} & 0 \\ 0 & C P_{n+1|n} C^T + R \end{pmatrix}\right)$$

Note:

$$\text{Cov}(Y_{n+1|n} | X_{n+1|n}) = \text{Cov}(C X_{n+1|n} + V_{n+1|n}, X_{n+1|n})$$

$$= C \text{Cov}(X_{n+1|n}, X_{n+1|n}) + \text{Cov}(V_{n+1|n}, X_{n+1|n})$$

$$= C \text{Var}(X_{n+1|n}) = C P_{n+1|n}$$
8. Recall that

\[ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \mathcal{N} \left( \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{pmatrix} \right) \]

\[ \mathbf{z}_1 | \mathbf{z}_2 \sim \mathcal{N} \left( \mathbf{m}_1 + \mathbf{z}_{12} (\mathbf{z}_2 - \mathbf{m}_2), \mathbf{z}_{11} - \mathbf{z}_{12} \mathbf{z}_{22} \right) \]

Compute:

\[ X_{n+1} | n+1 \sim \mathcal{N} \left( \mathbf{X}_{n+1} | n+1 = (X_{n+1} | n, Y_{n+1} | n) \right) \]

\[ \mathbf{X}_{n+1} | n+1 = \mathcal{N} \left( \begin{pmatrix} \mathbf{X}^\top_{n+1} | n + \mathbf{P}_{n+1} | n \left( \mathbf{C} \mathbf{P}_{n+1} | n \mathbf{C}^\top + \mathbf{R} \right)^{-1} (\mathbf{Y}_{n+1} - \mathbf{C} \mathbf{X}^\top_{n+1} | n) \\ \mathbf{P}_{n+1} | n \mathbf{C}^\top \left( \mathbf{C} \mathbf{P}_{n+1} | n \mathbf{C}^\top + \mathbf{R} \right)^{-1} \mathbf{C} \mathbf{P}_{n+1} | n \right) \right) \]

Now, let

\[ K_{n+1} = \mathbf{P}_{n+1} | n \mathbf{C}^\top \left( \mathbf{C} \mathbf{P}_{n+1} | n \mathbf{C}^\top + \mathbf{R} \right)^{-1} \]
\[ x_{n+1|n+1} = \hat{x}_{n+1|n} + k_{n+1} (y_{n+1} - C \hat{x}_{n+1|n}) \]

\[ p_{n+1|n+1} = p_{n+1|n} - k_{n+1} C p_{n+1|n} \]

Algorithm:

- Prediction step:
  \[ \hat{x}_{n+1|n} = A \hat{x}_{n|n} \]
  \[ p_{n+1|n} = A p_{n|n} A^T + Q \]

- Correction step:
  \[ k_{n+1} = p_{n+1|n} C^T (C p_{n+1|n} C^T + R)^{-1} \]
  \[ \hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + k_{n+1} (y_{n+1} - C \hat{x}_{n+1|n}) \]
  \[ p_{n+1|n+1} = p_{n+1|n} - k_{n+1} C p_{n+1|n} \]
Sequence Synthesis

Given $S_{xx}(\omega)$, can you generate $x[n]$ that has PSD $S_{xx}(\omega)$?

- White noise goal is to find $h[n]$

- $S_{ww}(\omega)$
  - $S_{xx}(\omega) = ?$
  - $S_{xx}(z) = H(z) H^*(\frac{1}{z}) \sigma_w^2$

To find $h[n]$, we first factor $S_{xx}(z)$ into $H(z)$ and $H^*(\frac{1}{z})$, then perform a $z$-inverse of $H(z)$ to find $h[n]$. 

$$R_{ww} = \sigma_w^2 S_{xx}[n].$$

$$E[w[n]x[n]] = 0.$$

$$S(\omega) = \frac{\sigma_w^2}{2\pi}.$$

$$R_{xx}[m] = h[m] * h[-m] * R_{ww}[m].$$
The function $S_{xx}(2)$ is called power spectral factorization.

Example:

$$S_{xx}(\omega) = \frac{5 + 4\cos 2\omega}{10 + 6\cos \omega}$$

$$S_{xx}(\omega) = 5 + 4 \left( \frac{e^{j2\omega} + e^{-j2\omega}}{2} \right)$$

$$= \frac{5 + 2e^{j2\omega} + 2e^{-j2\omega}}{10 + 3e^{j\omega} + 3e^{-j\omega}}$$

$$= \frac{5 + 2z^2 + 2z^{-2}}{10 + 3z + 3z^{-1}}$$
\[ \frac{2}{3} \left( 1 + \frac{1}{2} z^{-1} \right)^{-2} \]

\[ \frac{2}{3} \left( 1 + \frac{1}{3} z^{-1} \right) \]

\[
\begin{align*}
H(z) &= \frac{2}{3} \left( 1 + \frac{1}{2} z^{-1} \right)^{-2} \\
&= \frac{2}{3} \left( 1 + \frac{1}{3} z^{-1} \right)
\end{align*}
\]

Choose \( H(z) = \frac{2}{3} \left( 1 + \frac{1}{3} z^{-1} \right)^{-2} \)

Because it is causal and stable, \( n - 2 \)

\[
\begin{align*}
h[n] &= \frac{2}{3} \left( -\frac{1}{3} \right)^n \nu[n] + \frac{2}{3} \left( -\frac{1}{3} \right)^{n-2} \nu[n-2]
\end{align*}
\]
Channel Equalization

\[ x[n] \rightarrow [h(n)] \rightarrow [y[n]] \rightarrow [\frac{1}{H(z)}] \rightarrow x[n] \]

White noise

distorted

Use power spectral factorization of \( S_{yy}(\omega) \)

\[ S_{yy}(\omega) = 10 + 6 \cos \omega \]

\[ = 10 + 6e^{j\omega} + 6e^{-j\omega} \]

\[ \Rightarrow H(z) = 10 + 3z + 3z^{-1} \]

\[ Y(z) = 3(1 + \frac{1}{z}) \quad 3(1 + \frac{1}{z}) \]
\[ z = \left( \frac{m^2}{z^2 + 1} \right)^2 = (2\ell + 1/2) \]