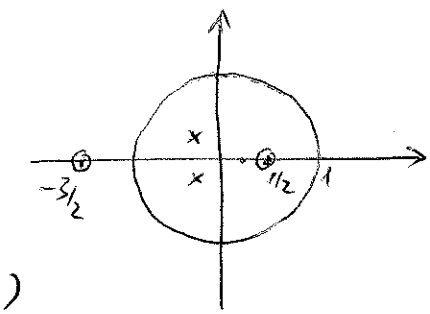
HW 3 Solution

① a/ $$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + z^{-1} - 0.75 z^{-2}}{1 + \frac{1}{2\sqrt{2}} z^{-1} + \frac{1}{16} z^{-2}}$$

$$\Rightarrow \quad y[n] + \frac{1}{2\sqrt{2}} y[n-1] + \frac{1}{16} y[n-2] = x[n] + x[n-1] - 0.75\, x[n-2]$$

b/ zeros: $\frac{1}{2}, -\frac{3}{2}$

poles: $\frac{1}{4} e^{-j\frac{3\pi}{4}}, \ \frac{1}{4} e^{j\frac{3\pi}{4}}$



ROC: $|z| > \frac{1}{4}$ (since $H(z)$ is stable)

c/ $$|H(e^{j\omega})|_{dB} = \sum_{i=1}^{4} |H_i(e^{j\omega})|_{dB}$$

* zeros:

+) $r = \frac{1}{2}, \ \theta = 0$: $$|H_1(e^{j\omega})|_{dB} = 10 \log_{10}(1 + r^2 - 2r\cos(\omega - \theta)) = 10 \log_{10}\left(\frac{5}{4} - \cos\omega\right)$$

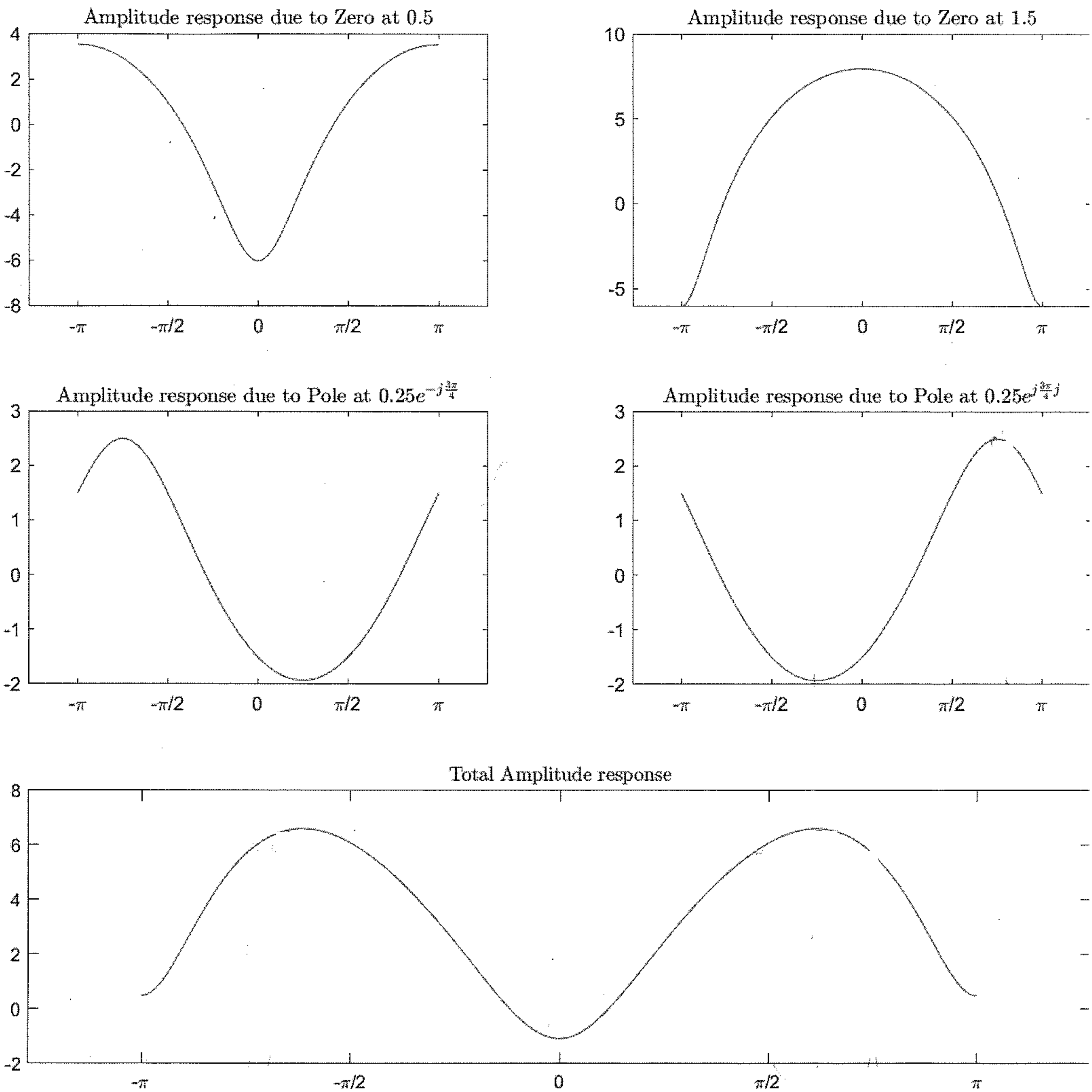
+) $r = \frac{3}{2}, \ \theta = \pi$: $$|H_2(e^{j\omega})|_{dB} = 10 \log_{10}\left(\frac{13}{9} + 3\cos\omega\right)$$

* poles:

+) $r = \frac{1}{4}, \ \theta = -\frac{3\pi}{4}$: $$|H_3(e^{j\omega})|_{dB} = -10 \log_{10}\left(\frac{17}{16} - \frac{1}{2}\cos\left(\omega + \frac{3\pi}{4}\right)\right)$$

+) $r = \frac{1}{4}, \ \theta = \frac{3\pi}{4}$: $$|H_4(e^{j\omega})|_{dB} = -10 \log_{10}\left(\frac{17}{16} - \frac{1}{2}\cos\left(\omega - \frac{3\pi}{4}\right)\right)$$

(See Fig. 1)



Amplitude response due to Zero at 0.5
Amplitude response due to Zero at 1.5
Amplitude response due to Pole at $0.25e^{-j\frac{3\pi}{4}}$
Amplitude response due to Pole at $0.25e^{j\frac{3\pi}{4}j}$
Total Amplitude response
$-\pi$
$-\pi/2$
0
$\pi/2$
π

Figure 1.

d/

I. True. ROC is outside all poles

II. False. See the plot.

III. False. Zeros of the original system are poles of the inverse system $\Rightarrow$ the zero at $z = -3/2$ makes the inverse system cannot be stable and causal.

IV. ~~True~~ True. They are: $|z| > \frac{3}{2}$; $\frac{1}{2} < |z| < \frac{3}{2}$; $\frac{1}{4} < |z| < \frac{1}{2}$

(2)

Zeros: $r = 0.8$, $\theta = \pm \frac{\pi}{3}$

poles: $r = 0.95$, $\theta = \pm \frac{2\pi}{3}$

a/ $|H(e^{j\omega})|_{dB} = \sum_{i=1}^{4} |H_i(e^{j\omega})|_{dB}$

* zeros:

+) $r = 0.8$, $\theta = \frac{\pi}{3}$: $|H_1(e^{j\omega})|_{dB} = 10 \log_{10}(1.64 - 1.6 \cos(\omega - \frac{\pi}{3}))$

+) $r = 0.8$, $\theta = -\frac{\pi}{3}$: $|H_2(e^{j\omega})|_{dB} = 10 \log_{10}(1.64 - 1.6 \cos(\omega + \frac{\pi}{3}))$

* poles:

+) $r = 0.95$, $\theta = \frac{2\pi}{3}$: $|H_3(e^{j\omega})|_{dB} = -10 \log_{10}(1.9025 - 1.9 \cos(\omega - \frac{2\pi}{3}))$

+) $r = 0.95$, $\theta = -\frac{2\pi}{3}$: $|H_4(e^{j\omega})|_{dB} = -10 \log_{10}(1.9025 - 1.9 \cos(\omega + \frac{2\pi}{3}))$

(see Fig. 2)

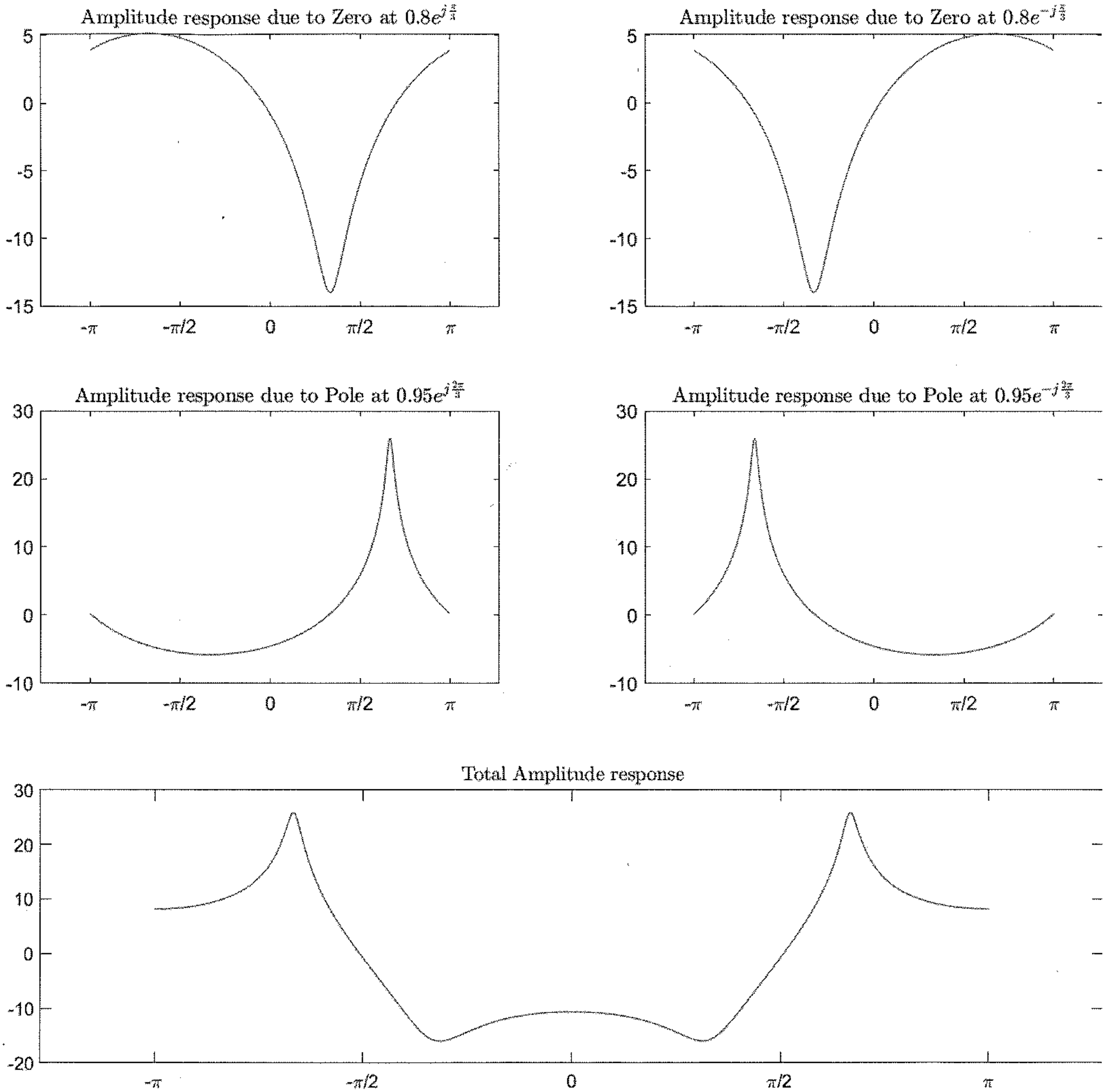

Amplitude response due to Zero at $0.8e^{j\frac{\pi}{3}}$
Amplitude response due to Zero at $0.8e^{-j\frac{\pi}{3}}$
Amplitude response due to Pole at $0.95e^{j\frac{2\pi}{3}}$
Amplitude response due to Pole at $0.95e^{-j\frac{2\pi}{3}}$
Total Amplitude response
5
0
-5
-10
-15
30
20
10
0
-10
-20
$-\pi$
$-\pi/2$
0
$\pi/2$
π

Figure 2.

b/ $\quad grd(H(e^{j\omega})) = \sum_{i=1}^{4} grd_i(H_i(e^{j\omega}))$

* zeros:

+) $r = 0.8,\ \theta = \frac{\pi}{3}$: $\quad grd_1(H_1(e^{j\omega})) = \dfrac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)}$

$$= \frac{0.64 - 0.8\cos(\omega - \frac{\pi}{3})}{1.64 - 1.6\cos(\omega - \frac{\pi}{3})}$$

+) $r = 0.8,\ \theta = -\frac{\pi}{3}$: $\quad grd(H_2(e^{j\omega})) = \dfrac{0.64 - 0.8\cos(\omega + \frac{\pi}{3})}{1.64 - 1.6\cos(\omega + \frac{\pi}{3})}$

* poles:

+) $r = 0.95,\ \theta = \frac{2\pi}{3}$: $\quad grd(H_3(e^{j\omega})) = -\dfrac{0.9025 - 0.95\cos(\omega - \frac{2\pi}{3})}{1.9025 - 1.9\cos(\omega - \frac{2\pi}{3})}$

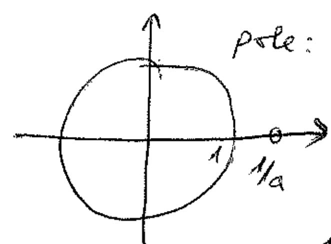
+) $r = 0.95,\ \theta = -\frac{2\pi}{3}$: $\quad grd(H_4(e^{j\omega})) = -\dfrac{0.9025 - 0.95\cos(\omega + \frac{2\pi}{3})}{1.9025 - 0.95\cos(\omega + \frac{2\pi}{3})}$

(see Fig. 3)

3 a/ $\quad H(e^{j\omega}) = e^{j\omega} - \frac{1}{a} = \cos\omega + j\sin\omega - \frac{1}{a}$

$\rightarrow \measuredangle H(e^{j\omega}) = \arctan\left(\dfrac{\sin\omega}{\cos\omega - \frac{1}{a}}\right)$

zero: $\frac{1}{a}$
pole: ∞



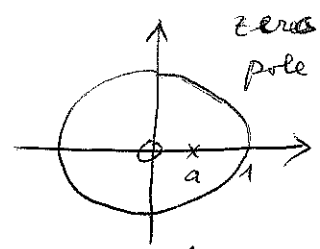
$H(z) = z - \frac{1}{a} = \dfrac{1 - \frac{1}{a}z^{-1}}{z^{-1}}$

b/ $\quad G(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}} = \dfrac{1}{1 - a\cos\omega + ja\sin\omega}$

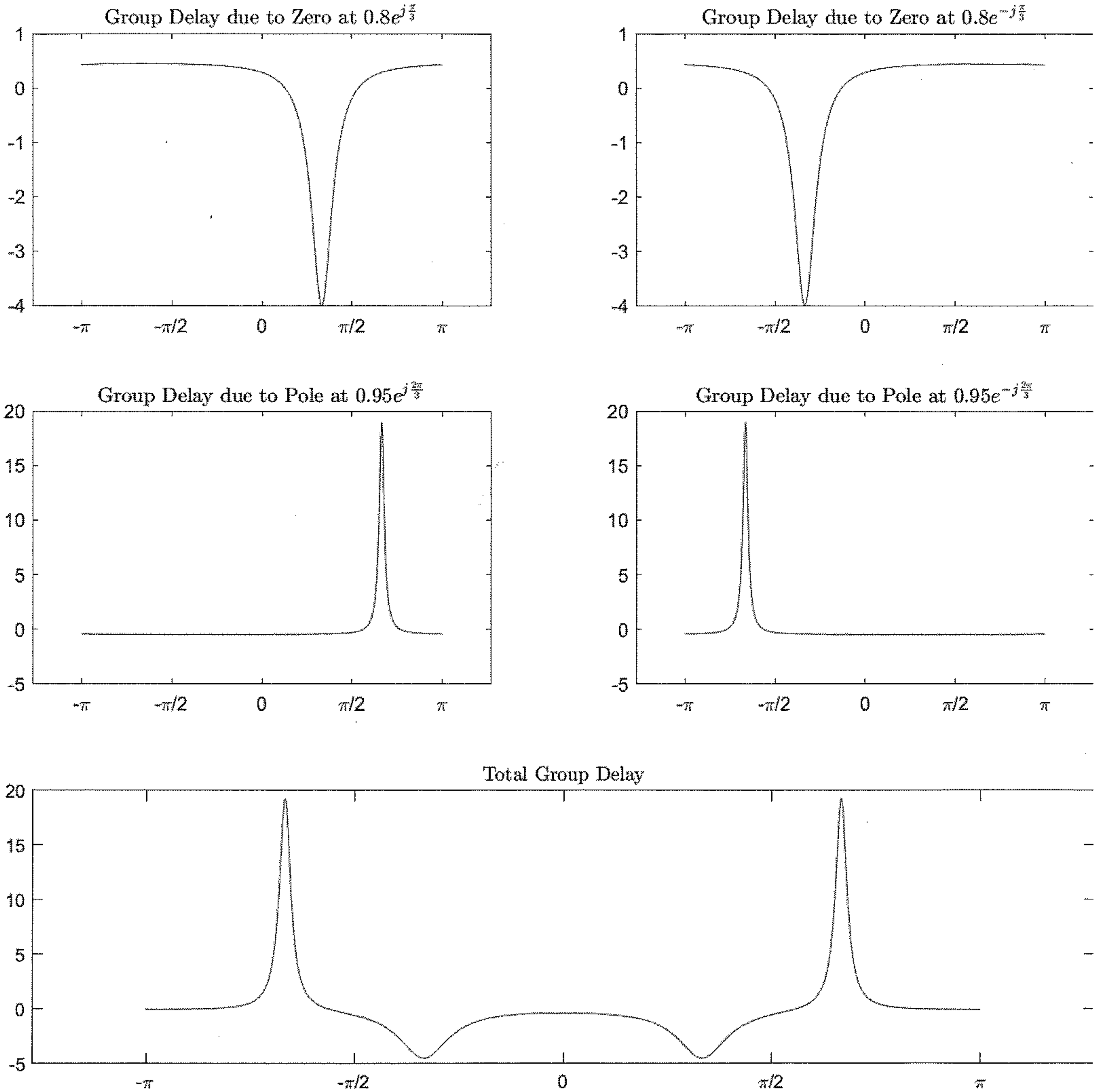
$$= \frac{1 - a\cos\omega - ja\sin\omega}{(1 - a\cos\omega)^2 + (a\sin\omega)^2}$$

$\Rightarrow \measuredangle G(e^{j\omega}) = \arctan\left(\dfrac{-a\sin\omega}{1 - a\cos\omega}\right) = \arctan\left(\dfrac{\sin\omega}{\cos\omega - \frac{1}{a}}\right)$

zeros: 0
pole: a



$H(z) = \dfrac{1}{1 - az^{-1}}$



Group Delay due to Zero at $0.8e^{j\frac{\pi}{3}}$
Group Delay due to Zero at $0.8e^{-j\frac{\pi}{3}}$
1
0
-1
-2
-3
-4
Group Delay due to Pole at $0.95e^{j\frac{2\pi}{3}}$
Group Delay due to Pole at $0.95e^{-j\frac{2\pi}{3}}$
20
15
10
5
0
-5
Total Group Delay
$-\pi$
$-\pi/2$
0
$\pi/2$
π

Figure 3.

(4) For an all-pass system:

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

All poles and zeros are in conjugate reciprocal pairs (CRP)

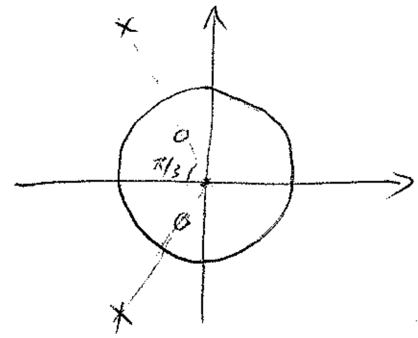
a/ No. $H_1(z)$ doesn't have pole-zero in CRP.

b/ Yes. $H_2(z)$ has $c_1 = -1-j = \sqrt{2} e^{-j 3\pi/4}$

c/ Yes. $H_3(z)$ has $d_1 = 0$ and $c_1 = -\frac{4}{3} j = -\frac{4}{3} e^{-j \pi/2}$

d/ No. $H_4(z)$ has $d_1 = \infty$ but the pole-zero is not in CRP.

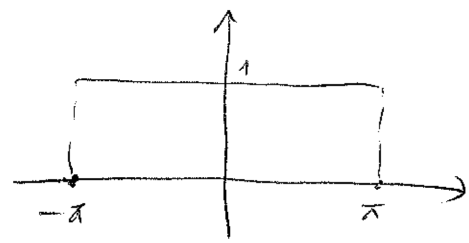
(5) $c = -1 + \sqrt{3}\, j = 2 e^{j 2\pi/3}$



a/ zeros: $r = \frac{1}{2}$, $\theta = \pm \frac{2\pi}{3}$

poles: $r = 2$, $\theta = \pm \frac{2\pi}{3}$

b/ $|H(e^{j\omega})| = 1$ since $H(z)$ is an all-pass system

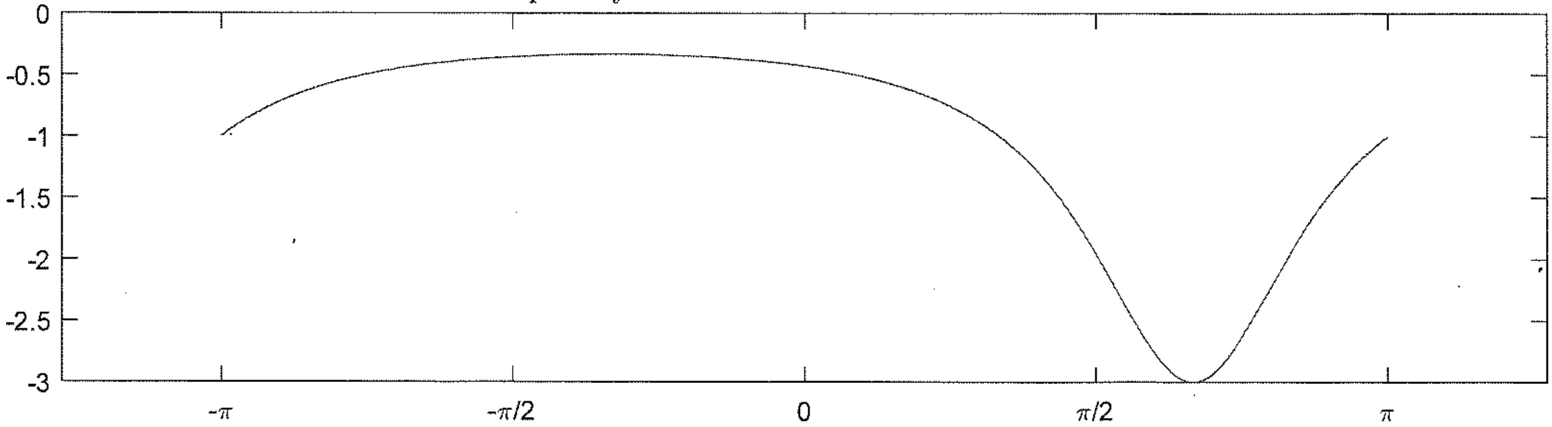


c/

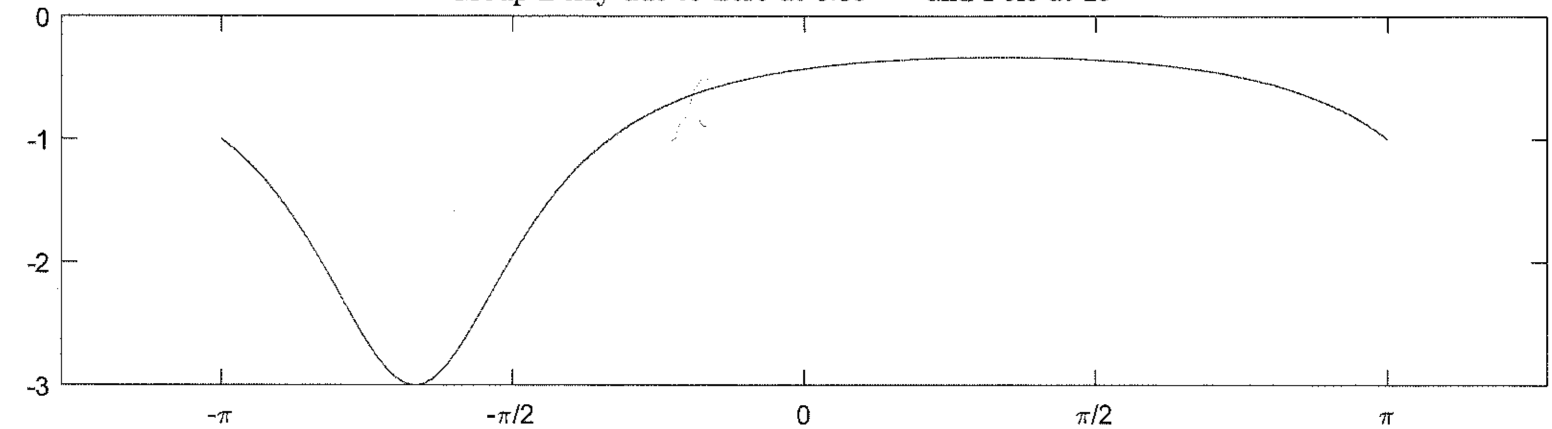
$$\operatorname{grd}(H(e^{j\omega})) = \frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)} + \frac{1-r^2}{1+r^2-2r(\cos\omega+\theta)}$$

$$= \frac{-3}{5-4\cos(\omega-\frac{2\pi}{3})} + \frac{-3}{5-4\cos(\omega+\frac{2\pi}{3})}$$

Group Delay due to Zero at $0.5e^{j\frac{2\pi}{3}}$ and Pole at $2e^{j\frac{2\pi}{3}}$



Group Delay due to Zero at $0.5e^{-j\frac{2\pi}{3}}$ and Pole at $2e^{-j\frac{2\pi}{3}}$



Total Group Delay