HW5 Solution

(1)

a/ $h_1[n] = \delta[n-1] + \delta[n-4]$

$$\Rightarrow H(e^{j\omega}) = e^{-j\omega} + e^{-4j\omega} = e^{-\frac{5}{2}j\omega}\left(e^{\frac{3}{2}j\omega} + e^{-\frac{3}{2}j\omega}\right)$$

$$= e^{-\frac{5}{2}j\omega} \cdot 2\cos\left(\frac{3}{2}\omega\right)$$

- This is not a linear-phase (LP) filter since $2\cos\left(\frac{3}{2}\omega\right)$ can be negative (e.g., for $\omega \in \left(\frac{\pi}{3}, \pi\right)$).

- This is a generalized linear-phase (GLP) filter with

$$A(e^{j\omega}) = 2\cos\left(\frac{3\omega}{2}\right) \; ; \quad \alpha = \frac{5}{2} \; ; \quad \beta = 0.$$

b/ $h_2[n] = -2\delta[n] + \delta[n-3]$

This is neither symmetric nor anti-symmetric, so it is not GLP/LP.

c/ $h_3[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - 2\delta[n-4] + \delta[n-5] - \delta[n-6]$

$$\Rightarrow H(e^{j\omega}) = 1 - e^{-j\omega} + 2e^{-2j\omega} - 2e^{-4j\omega} + e^{-5j\omega} - e^{-6j\omega}$$

$$= e^{-3j\omega}\left(e^{3j\omega} - e^{2j\omega} + 2e^{j\omega} - 2e^{-j\omega} + e^{-2j\omega} - e^{-3j\omega}\right)$$

$$= e^{-3j\omega}\left(2j\sin 3\omega - 2j\sin 2\omega + 4j\sin\omega\right)$$

$$= e^{-3j\omega + j\frac{\pi}{2}} \cdot 2\left(\sin 3\omega - \sin 2\omega + 2\sin\omega\right)$$

- This is not LP since $2(\sin 3\omega - \sin 2\omega + 2\sin\omega)$ can be negative (e.g., for $\omega \in [-\pi, 0]$).

- This is GLP with $A(e^{j\omega}) = 2(\sin 3\omega - \sin 2\omega + 2\sin\omega)$

$\alpha = 3$ and $\beta = \frac{\pi}{2}$.

d/ $h_4[n] = \delta[n] + 2\delta[n-1] + 4\delta[n-2] + 2\delta[n-3] + \delta[n-4]$

$$\Rightarrow H(e^{j\omega}) = 1 + 2e^{-j\omega} + 4e^{-2j\omega} + 2e^{-3j\omega} + e^{-4j\omega}$$

$$= e^{-2j\omega}\left(e^{2j\omega} + 2e^{j\omega} + 4 + 2e^{-j\omega} + e^{-2j\omega}\right)$$

$$= e^{-2j\omega}\left(4 + \underbrace{2\cos 2\omega}_{2\cos^2\omega - 1} + 4\cos\omega\right)$$

$$= e^{-2j\omega}\left(4(\cos\omega)^2 + 4\cos\omega + 2\right)$$

$$= e^{-2j\omega}\left((2\cos\omega + 1)^2 + 1\right)$$

- This is a LP with $\begin{cases} |H(e^{j\omega})| = 1 + (2\cos\omega + 1)^2 > 0 \\ \alpha = 2 \end{cases}$

- This is also a GLP with $\begin{cases} A(e^{j\omega}) = 1 + (2\cos\omega + 1)^2 \\ \alpha = 2 \ ; \ \beta = 0 \end{cases}$

e/ $h_5[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3]$

This is neither symmetric nor anti-symmetric,

so it is not GLP/LP.

f/ $h_6[n] = \sum_{k=1}^{6} \delta[n-k]$

$$\Rightarrow H(e^{j\omega}) = \sum_{k=1}^{6} e^{-j\omega k} = e^{-j\omega} \sum_{k=0}^{5} e^{-j\omega k}$$

$$= e^{-j\omega} \frac{1-e^{-6j\omega}}{1-e^{-j\omega}} = e^{-j\omega} \frac{2\sin 3\omega}{2\sin\frac{\omega}{2}}$$

$$= e^{-j\omega} \frac{\sin 3\omega}{\sin\frac{\omega}{2}}$$

- This is not LP since $\frac{\sin 3\omega}{\sin\frac{\omega}{2}}$ can be negative

(e.g. for $\omega \in (\frac{\pi}{3}, \frac{2\pi}{3})$)

- This is GLP with $\begin{cases} A(e^{j\omega}) = \dfrac{\sin 3\omega}{\sin\frac{\omega}{2}} \\ \alpha = 1;\ \beta = 0 \end{cases}$

(2)

a/ A, B, C, F

IIR systems have at least one pole that is non-zero.

b/ D, E

FIR systems have all poles at 0.

c/ B, C, D, E, F

Causal systems are stable if all poles lie inside the unit circle.

d/ F

Minimum-phase systems have all poles/zero inside the unit circle.

e/ D, E

GLP systems have poles at 0 and zeros come in conjugate reciprocal pairs.

f/ B

All-pass systems have pole-zero pairs in conjugate reciprocal.

g/ F

The stable and causal inverse system exists if all poles/zeros are inside the unit circle.

h/ D

Among two FIR systems D and E, we have the length of the impulse response is the number of zeros plus 1.

$\Rightarrow$ D has length 7 while E has length 11.

3) a/ The system has 5 zeros $\Rightarrow M = 5$

The length of the impulse response is $M+1 = 6$.

b/ For LP FIR systems, group delay is $\frac{M}{2} = 2.5$.

c/ For zero at $0.5e^{j\pi/4}$, we need another zero at its complex reciprocal $\Rightarrow a = 2e^{-j\pi/4}$

Since $h[n]$ is real, we also need a zero-pair at the complex conjugate positions: $\begin{cases} b = 0.5e^{-j\pi/4} \\ c = 2e^{j\pi/4} \end{cases}$

Finally, $H(e^{j\omega}) = 0$ at $\omega = 0$ implies an zero at $z=1$.

d/ $H(z) = (1-z^{-1})(1-2e^{-j\pi/4}z^{-1})(1-0.5e^{-j\pi/4}z^{-1})(1-2e^{j\pi/4}z^{-1})(1-0.5e^{j\pi/4}z^{-1})$

$= (1-z^{-1})(1-2\sqrt{2}z^{-1}+4z^{-2})(1-\frac{1}{\sqrt{2}}z^{-1}+\frac{1}{4}z^{-2})$

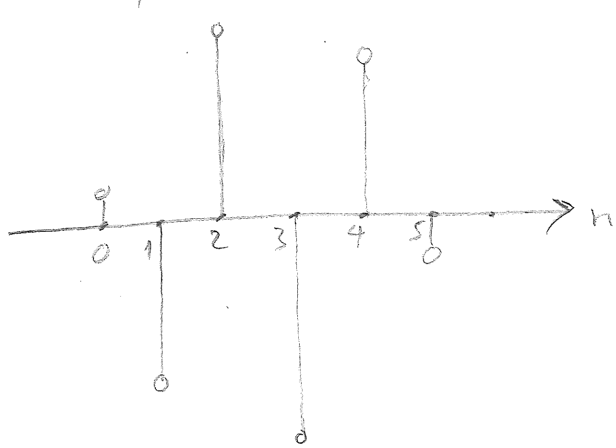
$= (1-z^{-1})(1-(2\sqrt{2}+\frac{1}{\sqrt{2}})z^{-1}+\frac{9}{4}z^{-2}-(2\sqrt{2}+\frac{1}{\sqrt{2}})z^{-3}+z^{-4})$

$= 1-(1+2\sqrt{2}+\frac{1}{\sqrt{2}})z^{-1}+(\frac{9}{4}+2\sqrt{2}+\frac{1}{\sqrt{2}})z^{-2}-(\frac{9}{4}+2\sqrt{2}+\frac{1}{\sqrt{2}})z^{-3}$

$+(1+2\sqrt{2}+\frac{1}{\sqrt{2}})z^{-4}-z^{-5}$

$\Rightarrow h[n] = \delta[n]-(1+2\sqrt{2}+\frac{1}{\sqrt{2}})\delta[n-1]+(\frac{9}{4}+2\sqrt{2}+\frac{1}{\sqrt{2}})\delta[n-2]$

$-(\frac{9}{4}+2\sqrt{2}+\frac{1}{\sqrt{2}})\delta[n-3]+(1+2\sqrt{2}+\frac{1}{\sqrt{2}})\delta[n-4]-\delta[n-5]$



e/ $h[n] = -h[5-n]$ $\Rightarrow$ odd & anti-symmetry

Type IV system.

④ $\begin{cases} M = 10 \\ \text{anti-symmetric} \end{cases} \Rightarrow$ Type III $\Rightarrow$ zeros at -1 and 1

- zero at $-1+\sqrt{3}j \Rightarrow$ 3 other zeros at $\begin{cases} -1-\sqrt{3}j \\ -\frac{1}{4}+\frac{\sqrt{3}}{4}j \\ -\frac{1}{4}-\frac{\sqrt{3}}{4}j \end{cases}$

- zero at $\frac{1}{2}(1+j) \Rightarrow$ 3 other zeros at $\begin{cases} \frac{1}{2}(1-j) \\ (1+j) \\ (1-j) \end{cases}$

Thus,

$$H(z) = (1-z^{-1})(1+z^{-1})(1-(-1+\sqrt{3}j)z^{-1})(1-(-1+\sqrt{3}j)z^{-1})(1-(-\tfrac{1}{4}+\tfrac{\sqrt{3}}{4}j)z^{-1})$$
$$(1-(-\tfrac{1}{4}-\tfrac{\sqrt{3}}{4}j)z^{-1})(1-\tfrac{1}{2}(1+j)z^{-1})(1-\tfrac{1}{2}(1-j)z^{-1})(1-(1+j)z^{-1})(1-(1-j)z^{-1})$$

⑤ $$H(z) = \frac{2(1+4z^{-1})(1-0.25z^{-1})}{(4-(1-\sqrt{3}j)z^{-1})(1+0.5z^{-1})}$$

$$= \underbrace{\tfrac{1}{2}(1+4z^{-1})(1+0.25z^{-1})}_{H_{lin}} \cdot \underbrace{\frac{1-0.25z^{-1}}{(1-(\frac{1}{4}-\frac{\sqrt{3}}{4}j)z^{-1})(1+0.25z^{-1})(1+0.5z^{-1})}}_{H_{min}}$$