1 a/

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

(since $|\alpha| < 1 \Rightarrow |\alpha e^{-j\omega}| < 1$)

b/

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} = \sum_{n=0}^{N-1} \sum_{r=-\infty}^{\infty} x[n+rN] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \alpha^{n+rN} W_N^{kn} = \sum_{r=0}^{\infty} \sum_{n=0}^{N-1} \alpha^{rN} \left(\alpha W_N^{k}\right)^n$$

$$= \sum_{r=0}^{\infty} \alpha^{rN} \left(\sum_{n=0}^{N-1} \left(\alpha e^{-j\frac{2k\pi}{N}}\right)^n \right) = \left(\sum_{r=0}^{\infty} (\alpha^N)^r \right) \cdot \frac{1-\left(\alpha e^{-j\frac{2k\pi}{N}}\right)^N}{1-\alpha e^{-j\frac{2k\pi}{N}}}$$

$$= \frac{1}{1-\alpha^N} \cdot \frac{1-\alpha^N e^{-j2k\pi}}{1-\alpha e^{-j\frac{2k\pi}{N}}}$$

$e^{-j2k\pi} = 1$

(since $|\alpha| < 1 \Rightarrow |\alpha^N| < 1$)

$$= \frac{1}{1-\alpha e^{-j\frac{2k\pi}{N}}}$$

c/ From (a) and (b), we can see that

$$\tilde{X}[k] = X(e^{j\omega}) \text{ when } \omega = \frac{2k\pi}{N}$$

2 For $0 \le k \le N-1$:

a/ $x[n] = \delta[n]$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} = \sum_{n=0}^{N-1} \delta[n]\, W_N^{kn} = W_N^{0} = 1$$

b/ $x[n] = \delta[n-n_0]$, $0 \le n_0 \le N-1$

$$X[k] = \sum_{n=0}^{N-1} \delta[n-n_0]\, W_N^{kn} = W_N^{kn_0} = e^{-j\frac{2k\pi n_0}{N}}$$

c/ $x[n] = \begin{cases} 1 & n \text{ even} \quad 0 \le n \le N-1 \\ 0 & n \text{ odd} \quad 0 \le n \le N-1 \end{cases}$ (N even)

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} W_N^{2kn} = \sum_{n=0}^{\frac{N}{2}-1} \left(e^{-j\frac{4k\pi}{N}}\right)^n$$

$\left(e^{-j\frac{4k\pi}{N}} = 1 \iff \frac{2k}{N} \text{ is an integer} \iff \left[\begin{array}{l} k=0 \\ k=\frac{N}{2} \end{array}\right. \right)$

+) For $k=0$ or $k=\frac{N}{2} \Rightarrow e^{-j\frac{4k\pi}{N}} = 1$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} 1 = \frac{N}{2}$$

+) For other values of k :

$$X[k] = \frac{1 - e^{-j2k\pi}}{1 - e^{-j\frac{4k\pi}{N}}} = \frac{1-1}{1 - e^{-j\frac{4k\pi}{N}}} = 0$$

d/ $x[n] = \begin{cases} 1 & 0 \le n \le \frac{N}{2} - 1 \\ 0 & \frac{N}{2} - 1 \le n \le N-1 \end{cases}$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} \left(e^{-j\frac{2k\pi}{N}}\right)^n$$

+) For $k=0$: $X[k] = \sum_{n=0}^{\frac{N}{2}-1} 1 = \frac{N}{2}$

+) For $k \neq 0$: $X[k] = \dfrac{1 - e^{-jk\pi}}{1 - e^{-j\frac{2k\pi}{N}}} = \begin{cases} 0 & \text{if } k \text{ even} \\ \dfrac{2}{1-e^{-j\frac{2k\pi}{N}}} & \text{if } k \text{ odd} \end{cases}$

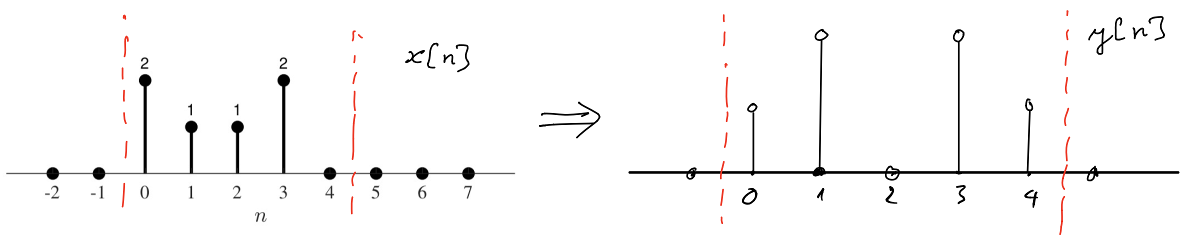
e/ $x[n] = \begin{cases} \alpha^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} \alpha^n W_N^{kn} = \frac{1 - (\alpha W_N^k)^N}{1 - \alpha W_N^k}$$

$$= \frac{1 - \alpha^N e^{-j2k\pi}}{1 - \alpha e^{-j\frac{2k\pi}{N}}}$$

(3) From the property of the DFT, $y[n]$ is equal to $x[n]$ circularly shifted by 2 to the left, i.e.,

$$y[((n))_5] = \tilde{y}[n] = \tilde{x}[n+2] = x[((n+2))_5]$$



(4) a/ $X(e^{j\omega}) = 6 + 5e^{-j\omega} + 4e^{-2j\omega} + 3e^{-3j\omega} + 2e^{-4j\omega} + e^{-5j\omega}$

b/ $$X[k] = \sum_{n=0}^{5} x[n]\, W_6^{kn} = 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$

where $W_6^{kn} = e^{-j\frac{2k\pi n}{6}} = e^{-j\frac{k\pi n}{3}}$

Hence:

$$X[0] = 6W_6^0 + 5W_6^0 + 4W_6^0 + 3W_6^0 + 2W_6^0 + W_6^0 = 21\, W_6^0 = 21$$

$$X[1] = 6W_6^0 + 5W_6^1 + 4W_6^2 + 3W_6^3 + 2W_6^4 + W_6^5$$

$$X[2] = 6W_6^0 + 5W_6^2 + 4W_6^4 + 3W_6^0 + 2W_6^2 + W_6^4 = 9W_6^0 + 7W_6^2 + 5W_6^4$$

$$X[3] = 6W_6^0 + 5W_6^3 + 4W_6^0 + 3W_6^3 + 2W_6^0 + W_6^3 = 12W_6^0 + 9W_6^3 = 3$$

$$X[4] = 6W_6^0 + 5W_6^4 + 4W_6^2 + 3W_6^0 + 2W_6^4 + W_6^2 = 9W_6^0 + 5W_6^2 + 7W_6^4$$

$$X[5] = 6W_6^0 + 5W_6^5 + 4W_6^4 + 3W_6^3 + 2W_6^2 + W_6^1$$

(Alternatively, for $k \neq 0$:

$$X[k] = \sum_{n=0}^{5} x[n] W_6^{kn} = \sum_{n=0}^{5} (6-n) W_6^{kn}$$

$$= 6 \sum_{n=0}^{5} W_6^{kn} - \sum_{n=0}^{5} n\, e^{-j \overbrace{\frac{2\pi k}{6}}^{\omega} n}$$

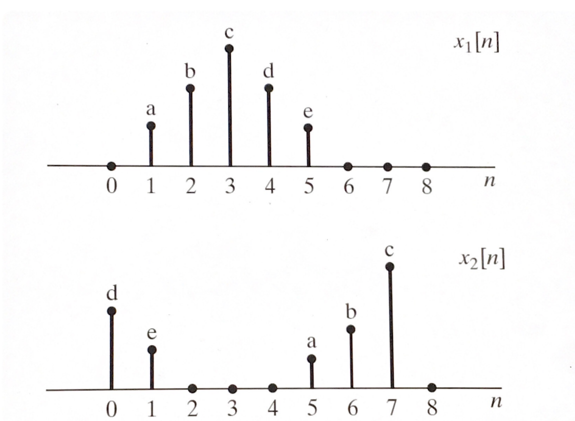
$$= 6 \frac{1 - W_6^{k\cdot 6}}{1 - W_6^{k}} - j \sum_{n=0}^{5} \frac{d}{d\omega} e^{-j\omega n}$$

$$= 0 - j \frac{d}{d\omega} \sum_{n=0}^{5} e^{-j\omega n} = -j \frac{d}{d\omega} \frac{1 - e^{-6j\omega}}{1 - e^{-j\omega}}$$

$$= -j \frac{6j e^{-6j\omega}(1 - e^{-j\omega}) - j e^{-j\omega}(1 - e^{-6j\omega})}{(1 - e^{-j\omega})^2} \qquad (e^{-6j\omega} = e^{-j2k\pi} = 1)$$

$$= -j \frac{6j(1 - e^{-j\omega}) - 0}{(1 - e^{-j\frac{2\pi k}{6}})^2} = \frac{6}{1 - e^{-j\frac{\pi k}{3}}} = \frac{-3j}{e^{-j\frac{\pi k}{6}} \sin(\frac{\pi k}{6})} = \frac{3}{\sin(\frac{\pi k}{6})} e^{j(\frac{\pi k}{6} - \frac{\pi}{2})}$$)

5



From the figure, it can be seen that the two sequences are related through a circle shift

$$x_2[((n))_8] = \tilde{x}_2[n] = \tilde{x}_1[n-4] = x_1[((n-4))_8]$$

$$\Rightarrow X_2[k] = W_8^{4k} X_1[k] = e^{-j\frac{2\pi \cdot 4k}{8}} X_1[k] = (-1)^k X_1[k]$$