ROB 538: Multiagent Systems

Week 5, Lecture 2:

Game Theory
(Based on E. Elkind and E. Markakis from Multagent Systems, Ed. Weiss)

Reading:
Chapter 17

Announcements:
Homework 3 due: 10/31
Preliminary paper due: 11/7
Midterm Exam: 11/9

What does Game Theory Study

• Interactions of rational decision-makers (agents, players)

• Decision-makers: humans, robots, computer programs, firms in the market, political parties

• Rational: each agent has preferences over outcomes and chooses an action that is most likely to lead to the best feasible outcome

• Interactions: 2 or more agents act simultaneously or consequently
Why Study Game Theory?

- To understand the behavior of others in strategic situations
- To know how to alter one’s own behavior in such situations to gain advantage
- Wikipedia: game theory attempts to mathematically capture behavior in strategic situations, in which an individual success in making choices depends on the choices of others

A Bit of History

- Early ideas:
  - Models on competition among firms: Cournot (=1838), Bertrand (=1883)
  - 0-sum games: end of 19th century (Zermelo) and early 20th century (Borel)
- Foundations of the field (1944):
  Theory of Games and Economic Behavior by
  John von Neumann and Oskar Morgenstern
- Key concept: Nash equilibrium
  (John Nash, 1951)
- Main applications:
  - microeconomics
  - political science
  - evolutionary biology
Normal-Form Games

- Complete-information games
  - players know each other’s preferences

- Simultaneous moves
  - All players choose their action at the same time (or at the time they make their own choice, they do not know or cannot observe the other players’ choices)

Formally:

- A normal-form game is given by
  - a set of players \( N \)
  - for each player \( i \), a set of available actions \( A_i \)
  - for each player \( i \), a utility function \( u_i : A_1 \times ... \times A_n \rightarrow \mathbb{R} \) (real numbers)

- Action profile: Any vector \( (a_1, ..., a_n) \), with \( a_i \in A_i \)
  - each action profile corresponds to an outcome
  - \( u_i \) describes how much player \( i \) enjoys each outcome
Utilities and Preferences

Prisoner’s Dilemma

- Two agents committed a crime.
- The court does not have enough evidence to convict them of the crime, but can convict them of a minor offence (1 year in prison each)
- If one suspect confesses (acts as an informer), he walks free, and the other suspect gets 4 years
- If both confess, each gets 3 years
- Agents have no way of communicating or making binding agreements
• Set of players $N = \{1, 2\}$
• $A_1 = A_2 = \{\text{confess (C), stay quiet (Q)}\}$
• $u_1(C, C) = -3$ (both get 3 years)
• $u_1(C, Q) = 0$ (player 1 walks free)
• $u_1(Q, C) = -4$ (player 1 gets 4 years)
• $u_1(Q, Q) = -1$ (both get 1 year)
• $u_2(x, y) = u_1(y, x)$

---

Prisoner’s Dilemma: Matrix Representation

<table>
<thead>
<tr>
<th></th>
<th>quiet</th>
<th>confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>quiet</td>
<td>$(-1, -1)$</td>
<td>$(-4, 0)$</td>
</tr>
<tr>
<td>confess</td>
<td>$(0, -4)$</td>
<td>$(-3, -3)$</td>
</tr>
</tbody>
</table>

Interpretation: the pair $(x, y)$ at the intersection of row $i$ and column $j$ means that the row player gets $x$ and the column player gets $y$.
Prisoner’s Dilemma: The Rational Outcome

• P1’s reasoning:
  - if P2 stays quiet, I should confess
  - if P2 confesses, I should confess, too

• P2 reasons in the same way
  - if P1 stays quiet, I should confess
  - if P1 confesses, I should confess, too

Result: both confess and get 3 years in prison

  – Note: if they chose to cooperate and stay quiet, they get away with 1 year each

Dominant Strategy: Definition

• **Dominant strategy**: a strategy that is best for a player no matter what the others choose

• **Definition**: a strategy a of player i is said to be a dominant strategy for i, if

\[ u_i(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \geq u_i(a_1, \ldots, a_{i-1}, a', a_{i+1}, \ldots, a_n) \]

• for any \( a' \in A_i \) and any strategies \( a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \) of other players.

• In Prisoner’s Dilemma, Confess is a **dominant strategy** for each of the players
Dominant Strategy: Discussion

- Can a player have more than one dominant strategy?
  - It can happen if some actions result always in the same utility

- Fact: each player has at most one strictly dominant strategy

- A rational agent will never play a dominated strategy.

- So in deciding what to do, we can delete dominated strategies.

The Joint Project Game

- Two students are assigned a project

- If at least one of them works hard, the project succeeds

- Each student
  - wants the project to succeed (+5)
  - prefers not to make an effort (-2)
  - hates to be exploited, i.e., work hard when the other slacks (-5)

\[
\begin{array}{c|cc}
& \text{Work} & \text{Slack} \\
\hline
\text{P1} & (3, 3) & (-5, 5) \\
\text{P2} & (5, -5) & (0, 0) \\
\end{array}
\]
Joint Project vs. Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>(3, 3)</td>
<td>(5, -5)</td>
</tr>
<tr>
<td>S</td>
<td>(5, -5)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

- In JP, row player prefers (S, W) to (W, W) to (S, S) to (W, S)
- In PD, row player prefers (C, Q) to (Q, Q) to (C, C) to (Q, C)
  - column player has similar preferences
  - These two games are equivalent!

Battle of Sexes

<table>
<thead>
<tr>
<th></th>
<th>Theatre</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theatre</td>
<td>(2, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Football</td>
<td>(0, 0)</td>
<td>(1, 2)</td>
</tr>
</tbody>
</table>

- Charlie and Marcie want to go out, either to theatre or to a football game
- She prefers theatre, he prefers football
- But they will be miserable if they go to different places
Battle of Sexes

• No player has a dominant strategy:
  - T is not a dominant strategy for Marcie: if Charlie chooses F, Marcie prefers F
  - F is not a dominant strategy for Marcie: if Charlie chooses T, Marcie prefers T

• However, (T, T) is a stable strategy:
  - neither player wants to change his or her action given the other player’s action

• (F, F) is stable, too

Notation

• Given a vector \( a = (a_1, ..., a_n) \), let \( (a_{-i}, a') \) be \( a \), but with \( a_i \) replaced by \( a' \):

\[
(a_{-i}, a') = (a_1, ..., a_{i-1}, a', a_{i+1}, ..., a_n)
\]

• If \( a = (3, 5, 7, 8) \), then \( (a_{-3}, 4) = (3, 5, 4, 8) \)
Nash Equilibrium (Nash’51)

• **Definition:** a strategy profile \( a = (a_1, ..., a_n) \)

  is a **Nash equilibrium (NE)** if no player can benefit by unilaterally changing her action:

  for each \( i = 1, ..., n \) :
  \[ u_i(a) \geq u_i(a_{-i}, a') \text{ for all } a' \in A_i \]

• 2 player case: \((a, b)\) is a NE if
  - \( u_1(a, b) \geq u_1(a', b) \) for every \( a' \in A_1 \)
  - \( u_2(a, b) \geq u_2(a, b') \) for every \( b' \in A_2 \)

---

Nash Equilibrium Pictorially

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>((x_1, _)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>((_, _)</td>
<td>((_, _)</td>
<td>((_, _)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((_, _)</td>
<td>((_, _)</td>
<td>((x_2, _)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((_, _)</td>
<td>((_, _)</td>
<td>((x_3, _)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((_, Y_1)</td>
<td>((_, Y_2)</td>
<td>((X, Y)</td>
<td>((_, Y_4)</td>
<td>((_, Y_5)</td>
<td></td>
</tr>
<tr>
<td>((_, _)</td>
<td>((_, _)</td>
<td>((x_5, _)</td>
<td>((_, _)</td>
<td>((_, _)</td>
<td></td>
</tr>
</tbody>
</table>

\( X \) must be at least as big as any \( x_i \) in \( Y \)-column
\( Y \) must be at least as big as any \( y_j \) in \( X \)-row
Nash Equilibria in Battle of Sexes

<table>
<thead>
<tr>
<th></th>
<th>Theatre</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Theatre</td>
<td>(2, 1)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>P1 Football</td>
<td>(0, 0)</td>
<td>(1, 2)</td>
</tr>
</tbody>
</table>

Both (T, T) and (F, F) are Nash equilibria

Nash Equilibrium and Dominant Strategies

- Prisoner’s dilemma: 
  (C, C) is a Nash equilibrium

<table>
<thead>
<tr>
<th></th>
<th>P2</th>
<th>Q</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Q</td>
<td>(-1, -1)</td>
<td>(-4, 0)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>(0, -4)</td>
<td>(-3, -3)</td>
<td></td>
</tr>
</tbody>
</table>

Theorem: In any 2-player normal-form game, if
- a is a dominant strategy for player 1, and
- b is a dominant strategy for player 2,
  then (a, b) is a Nash equilibrium
Best Response Functions

• Given a vector $a_i$ of other players’ actions, player $i$ may have one or more actions that maximize his utility

• Best response function:
  - $B_i(a_i) = \{a \in A_i \mid u_i(a_i, a) \geq u_i(a_i, a') \text{ for all } a' \in A_i\}$
  - $B_i(a_i)$ is set-valued
  - if $|B_i(a_i)| = 1$ for all $i$ and all $a_i$, we denote the single element of $B_i(a_i)$ by $b_i(a_i)$

Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(2*, 5*)</td>
<td>(3*, 3)</td>
<td>(6*, 3)</td>
</tr>
<tr>
<td>M</td>
<td>(2*, 7*)</td>
<td>(4*, 5)</td>
<td>(2*, 7*)</td>
</tr>
<tr>
<td>B</td>
<td>(1*, 4*)</td>
<td>(5*, 4*)</td>
<td>(2*, 1)</td>
</tr>
</tbody>
</table>

• $B_1(L) = \{T, M\}$
• $B_1(C) = \{B\}$
• $B_1(R) = \{T\}$

• $B_2(T) = \{L\}$
• $B_2(M) = \{L, R\}$
• $B_2(B) = \{L, C\}$
Best Responses and Nash Equilibria

• Recall:
  • \( a = (a_1, \ldots, a_n) \) is a Nash equilibrium if
    \[ u_i(a) \geq u_i(a_i, a') \]
    for all \( i \) and all \( a' \) in \( A_i \)

• In the language of best response functions:
  • \( a = (a_1, \ldots, a_n) \) is a Nash equilibrium if
    \( a_i \) is in \( B_i(a_{-i}) \) for all \( i \)

Example Revisited

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(2*,5*)</td>
<td>(3,3)</td>
<td>(6*,3)</td>
</tr>
<tr>
<td>M</td>
<td>(2*,7*)</td>
<td>(4,5)</td>
<td>(2,7*)</td>
</tr>
<tr>
<td>B</td>
<td>(1,4*)</td>
<td>(5*,4*)</td>
<td>(2,1)</td>
</tr>
</tbody>
</table>

• \( \{T, L\}, \{M, L\} \) and \( \{B, C\} \) are Nash equilibria
Nash Equilibrium: Caution

- The definition does not say that each game has a Nash equilibrium
  - some do not

- The definition does not say that Nash equilibrium is unique
  - some games have many Nash equilibria

- Nash equilibrium outcomes need not be strictly better than the alternatives, what matters is that they are not worse (to a deviation)

- Not all equilibria are equally good
  - they can differ both in individual utilities and in total welfare

Non-Existence of Nash Equilibrium: Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>Heads</th>
<th>Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>Tails</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
</tr>
</tbody>
</table>

- Two players have 1 coin each
- They simultaneously decide whether to display their coin with Heads or Tails facing up
- If the coins match, player 1 gets both coins
- Otherwise player 2 gets them
**Non-Existence of Nash Equilibrium: Matching Pennies**

![Matching Pennies Table](image)

Q: How would we play this game in practice?
A: Toss a coin

**Matching Pennies: Randomization**

<table>
<thead>
<tr>
<th></th>
<th>1/2</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>T</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
</tr>
</tbody>
</table>

- Main idea: players may be allowed to play non-deterministically
- Suppose column player plays
  - H with probability ½
  - T with probability ½
- If we play H, the outcome is
  - (H, H) w.p. 1/2 (+1);
  - (H, T) w.p. 1/2 (-1)
- If we play T, the outcome is
  - (T, H) w.p. 1/2 (-1);
  - (T, T) w.p. 1/2 (+1)

P[win]=P[loss]=1/2
E[utility] = 0
Matching Pennies: Randomization

<table>
<thead>
<tr>
<th></th>
<th>1/2</th>
<th>1/2</th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
<td>(H, H) w.p. p/2,</td>
<td>(T, H) w.p. (1-p)/2,</td>
</tr>
<tr>
<td>T</td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
<td>(H, T) w.p. p/2,</td>
<td>(T, T) w.p. (1-p)/2</td>
</tr>
</tbody>
</table>

If we play H w.p. p, T w.p. 1-p, we get

Pr [+1] = Pr [(H, H) or (T, T)] = 1/2
Pr [-1] = Pr [(H, T) or (T, H)] = 1/2

No matter what we do, P[win]=P[loss]=1/2

How Should We Play?

- Suppose we (the row player) are playing against an opponent who mixes evenly: (H w.p. 1/2, T w.p. 1/2)
- Any strategy gives the same chance of winning (1/2)
- However, if we play H, the opponent can switch to playing T and win all the time
- Same if we play T
- If we play any action w.p. p < 1/2, the opponent can switch to this action and win w.p. 1-p > 1/2
- Thus, the only sensible choice is for us to mix evenly, too
Mixed Strategies

- A **mixed strategy** of a player in a strategic game is a probability distribution over the player’s actions.
- If the set of actions is \{a_1, ..., a_r\}, a mixed strategy is a vector \( p = (p_1, ..., p_r) \), where
  \[ p_i \geq 0 \text{ for } i = 1, ..., r, \quad p_1 + ... + p_r = 1 \]
- \( p(a_i) \): probability that the player chooses action \( a_i \)
- Matching pennies: mixing evenly can be written as
  \( p = (1/2, 1/2) \) or \( p(H) = p(T) = 1/2 \) or \( 1/2 T + 1/2 H \)
- \( P = (p_1, ..., p_n) \): mixed strategy profile
- Pure strategy: assigns probability 1 to some action

Mixed Strategies and Payoffs

- Suppose each player chooses a mixed strategy
- How do they reason about their utilities?
- Utilities need to be computed before the choice of action is realized
  - before the coin lands
- Mixed strategies generate a probability space
- Players are interested in their expected utility w.r.t. this space
Expected Utility (2 Players)

- Player 1’s set of actions: \( A = \{a^1, ..., a^r\} \)
- Player 2’s set of actions: \( B = \{b^1, ..., b^s\} \)
- Player 1’s utility is given by \( u_1: A \times B \rightarrow R \)
- If player 1 plays mixed strategy \( p = (p^1, ..., p^r) \), and player 2 plays mixed strategy \( q = (q^1, ..., q^s) \)
- The expected utility of player 1 is
  \[
  U_1(p, q) = \sum_{i=1}^{r} \sum_{j=1}^{s} p^i q^j u_1(a^i, b^j)
  \]
- Similarly for player 2 (replace \( u_1 \) by \( u_2 \))

Equilibria in Mixed Strategies

- **Definition:** A mixed strategy profile \( P = (p_1, ..., p_n) \) is a mixed strategy Nash equilibrium if for any player \( i \) and any mixed strategy \( p' \) of player \( i \),
  \[
  U_i(P) \geq U_i(P'_{-i}, p')
  \]
- We refer to Nash equilibria in pure strategies as pure Nash equilibria
Equilibria in Mixed Strategies

• Theorem [Nash 1951]:

   Every n-player strategic game in which each player has a finite number of actions has at least one Nash equilibrium in mixed strategies

Solution Concepts

• How does a rational agent behave in a given scenario:

• Play
  - Dominant strategy
  - Nash Equilibrium strategy
  - Pareto Optimal strategy
  - Social welfare maximization strategy
Pareto Optimality

• An outcome is said to be Pareto optimal (or Pareto efficient) if there is no other outcome that makes one agent better off without making another agent worse off.

• If an outcome is Pareto optimal, then at least one agent will be reluctant to move away from it (because this agent will be worse off).

• If an outcome is not Pareto optimal, then there is another outcome that makes everyone as happy, if not happier. “Reasonable” agents would agree to move to in this case. (Even if I don’t directly benefit from, you can benefit without me suffering.)

Social Welfare

• The social welfare of an outcome $z$ is an aggregation of the utilities that each agent gets from outcome $z$. A possible social welfare function is:

$$ G = \sum_{i} u_i(z) $$

• It captures the “total amount of money in the system”

• As a solution concept, may be appropriate when the whole system (all agents) has a single owner (then overall benefit of the system is important, not individuals).
  - Robot coordination?
  - Traffic?
    • Not so much
Competitive and zero-sum interactions

• Where preferences of agents are in opposition we have strictly competitive scenarios.

• Zero-sum encounters are those where utilities sum to a zero:

\[ \sum_{i} u_i(z) = 0 \quad \forall z \in Z \]

(key feature is that sum is constant. Setting it to zero is for normalization)

• Zero sum implies strictly competitive.

Solution Concepts

• Nash equilibrium?

• Pareto efficiency?

• Social welfare?

<table>
<thead>
<tr>
<th></th>
<th>i defects</th>
<th>i cooperates</th>
</tr>
</thead>
<tbody>
<tr>
<td>j defects</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>j cooperates</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
Solution Concepts

• \((D,D)\) is the only Nash equilibrium.

• All outcomes except \((D,D)\) are Pareto optimal.

• \((C,C)\) maximizes social welfare.

\[
\begin{array}{c|cc|c}
 & i \text{ defects} & i \text{ cooperates} \\
\hline
j \text{ defects} & 2 & 5 & 0 \\
\hline
j \text{ cooperates} & 0 & 3 & 3 \\
\end{array}
\]
Extensive-Form Games

Simultaneous vs. Sequential Moves

• So far, we have considered games where players choose their strategies simultaneously.

  T or F?  T or F?

• What if players take turns choosing their actions?

  Football  OK, football
Games With Sequential Moves: More Examples

Chess and tic-tac-toe may differ in difficulty, but the underlying principle is the same: players take turns making moves, and eventually either one of the players wins or there is a tie.

Another Example: Market Entry

• Suppose that in some country firm 1 is currently the only available fast food chain
• Firm 2 considers opening their restaurants in that country
• Firm 2 has 2 actions: enter (E), stay out (S)
• If firm 2 stays out, firm 1 need not do anything
• If firm 2 enters, firm 1 can either fight (F) (lower prices, aggressive marketing) or accept (A)
Market Entry: Payoffs

- If firm 2 stays out, its payoff is 0, and firm 1 has a payoff of 2
- If firm 2 enters and firm 1 fights, each gets a payoff of -1
- If firm 2 enters and firm 1 accepts, they share the market, so both get a payoff of 1

Extensive Form Games: General Case

- An extensive-form game is described by a game tree:
  - rooted tree, with root corresponding to the start of the game
  - each internal node of the tree is labeled by a player
  - each leaf is labeled by a payoff vector (assigning a payoff to each player in the game)
  - For a node labeled by a player X, all edges leaving the node are labeled by the actions of player X
Market Entry: Predicting the Outcome

- How should players choose a strategy?
  - Firm 1 can reason as follows:
    - If firm 2 enters, the best for me is to play A
  - Firm 2 reasons as follows:
    - if I enter, firm 1 is better off accepting, so my payoff is 1
    - if I stay out, my payoff is 0
    - thus I am better off entering
- The only “rational” outcome is (E, A)
- Corresponds to a backward induction process

Predicting The Outcome: Backward Induction

- The outcome of the game can be predicted using backward induction:
  - Start with any node whose children are leaves only
  - For any such node, the agent who chooses the action will determine all payoffs including his own, so he will choose the action maximizing his payoff
    - breaking ties arbitrarily (we will come back to this)
  - Fix his choice of action, and delete other branches
  - Now his node has one outgoing edge, so it can be treated as a leaf
  - Repeat until the root’s action is determined
Predicting the Outcome: NE of the Normal-Form Game

• Can we use the (pure) Nash Equilibrium of a normal form game as a prediction for the outcome of G?

• **Claim:** any backward induction strategy profile in the extensive-form game corresponds to a Nash Equilibrium profile in the normal-form game.

• Is the reverse true?

Market Entry Revisited

• **Backward induction** outcome of the extensive-form game is \((E, A)\)

• **Nash equilibria** of the corresponding normal form game are \((E, A)\) and \((S, F)\)

• Thus, the converse is not true.
Classical Game Theory Limitations

- Perfect knowledge of the environment and payoff tables
- Perfectly rational agents
- Agents do not have frequency dependent responses
- Agents are unable to adapt

Questions?