Due: Tuesday, Nov. 20th

Integer Programming (IP)

In LP problems, the variables were required to be nonnegative.

i.e., \( x_j \geq 0 \)

⇒ Noninteger values for decision variables are impossible.

But many real-world problems require that

Some or all of the variables be integers.

Ex: Assignment of people, machines, etc. are expressed in integer quantities.

⇒ The problem must be solved, requiring that some or all of the variables are integers.

Such problems are called integer linear programming (ILP) problems.

There are basically 3 different types of ILPs:

1. Pure integer programming problem
   All variables are required to be integers
   i.e., \( x_j = 0, 1, 2, 3, \ldots \) for \( x_j \)

2. Mixed-integer programming (MIP) problem
   Some of the variables are required to be integers, while others are real variables.
1. \( x_j = 0, 1, 2, 3 \ldots \); for \( j = 1, 2 \ldots, p \)
\( x_j > 0 \); for \( j = p+1, p+2, \ldots, n \)

3. **Binary integer programming (BIP) problem**

1.e.) \( x_j = 0 \) or \( 1 \); \( j \)

The 1 here may mean the decision represented by variable \( x_j \) is ‘yes’, while 0 means that it is ‘no’.

**Examples**

1. **Either-or type constraints**

   We have 2 resources to choose from, and which one do we choose.

   In this case, we want only one of the two resource constraints to hold true.

\[
\begin{align*}
\text{either} & \quad 2x_1 + 5x_2 \leq 20 & \Rightarrow \text{Res. 1} \\
\text{or} & \quad 3x_1 + 2x_2 \leq 12 & \Rightarrow \text{Res. 2} \\
& \quad x_1, x_2 > 0 \\
\end{align*}
\]

This formulation is intractable.

\[
\begin{align*}
\text{either} & \quad 2x_1 + 5x_2 \leq 20 \\
& \quad \text{and} \quad 3x_1 + 2x_2 \leq 12 + M \\
\text{or} & \quad 2x_1 + 5x_2 \leq 20 + M \\
& \quad \text{and} \quad 3x_1 + 2x_2 \leq 12 \\
\end{align*}
\]
Also, since \( M \) is a very large positive number, both \( 3x_1 + 2x_2 \leq 12 + M \) and \( 2x_1 + 5x_2 \leq 20 + M \) are redundant constraints.

\[ 2x_1 + 5x_2 \leq 20 + yM \quad (1) \]
\[ 3x_1 + 2x_2 \leq 12 + (1-y)M \quad (2) \]

\[ 2x_1 + 5x_2 \leq 20 + (1-y)M \quad (3) \]
\[ 3x_1 + 2x_2 \leq 12 + yM \quad (4) \]

2. \( N \) out of the \( N \) constraints must hold true:

\[ f_1(x_1, x_2, \ldots, x_N) \leq d_1 \]
\[ f_2(x_1, x_2, \ldots, x_N) \leq d_2 \]
\[ \vdots \]
\[ f_N(x_1, x_2, \ldots, x_N) \leq d_N \]
Reformulate the problem as follows:

\[ f_1(x_1, x_2, \ldots, x_n) \leq d_1 + My_1 \]
\[ f_2(x_1, x_2, \ldots, x_n) \leq d_2 + My_2 \]

\[ \vdots \]

\[ f_N(x_1, x_2, \ldots, x_n) \leq d_N + My_N \]

and \[ \sum_{i=1}^{N} y_i = N - K \]

where \( y_i = 0 \) or \( 1 \); \( i = 1, 2, \ldots, N \)

3. Functions with \( N \) possible values

\[ f(x_1, x_2, \ldots, x_n) = d_1, d_2, d_3, \ldots, \text{ or } d_N. \]

\[ f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{N} d_i \cdot y_i \]

and \[ \sum_{i=1}^{N} y_i = 1 \]

Use of binary variables to represent general integer variables
If \( x \) is a general integer variable with bounds \( 0 \leq x \leq u \), where:
\[
\begin{align*}
N & = u - \frac{1}{2} \\
N+1 & < x < 2N \\
2 & \leq u < 2N
\end{align*}
\]
its binary representation is given by:
\[
x = \sum_{i=0}^{N} 2^i y_i
\]
where \( y_i = 0 \) or \( 1 \) are called the auxiliary binary variables.

Ex:
\[
3x_1 + 4x_2 \leq 20
\]
\[
x_2 \leq 5
\]
\[
x_1, x_2 \geq 0, \text{ and integer}
\]
\[
\Rightarrow \quad 0 \leq x_1 \leq 6
\]
for \( x_1 \), \[ 2 \leq 6 \leq 2^{2+1} \]
\[
\Rightarrow N = 2
\]
for \( x_2 \), \[ 2^2 \leq 5 \leq 2^{2+1} \]
\[
\Rightarrow N = 2
\]
\[
x_1 = \sum_{i=0}^{2} 2^i y_i = y_0 + 2y_1 + 4y_2
\]
\[
x_2 = \sum_{i=3}^{5} 2^i y_i = y_3 + 2y_4 + 4y_5
\]
The original constraints would now become:

\[ 3(y_0 + 2y_1 + 4y_2) + 4(y_3 + 2y_4 + 4y_5) \leq 20. \]

\[ y_3 + 2y_4 + 4y_5 \leq 5. \]

\[ 3y_0 + 6y_1 + 12y_2 + 4y_3 + 8y_4 + 16y_5 \leq 20. \]

\[ y_0, y_1, \ldots, y_5 = 0 \text{ or } 1. \]

**Capital budgeting problem**

Consider a problem where there are 5 projects, each having a life span of 3 yrs. The expenses and cumulative returns from each project are tabulated below:

<table>
<thead>
<tr>
<th>Project</th>
<th>Yr1</th>
<th>Yr2</th>
<th>Yr3</th>
<th>Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>2</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>10</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>8</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>8</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

All in thousands of dollars

The maximum amount available to spend each year is $24 K.
Max $Z = 30y_1 + 15y_2 + 40y_3 + 20y_4 + 20y_5$

S.t.

$7y_1 + 3y_2 + 8y_3 + 4y_4 + 5y_5 \leq 24$

$4y_1 + 9y_2 + 6y_3 + 7y_4 + y_5 \leq 24$

$y_1 + 2y_2 + 10y_3 + 10y_4 + 8y_5 \leq 24$

$y_j = 0 \text{ or } 1 \quad j = 1, 2, \ldots, 5$

$\rightarrow$ A BIP Problem