3. **Divisibility**
   
x\_j can take noninteger values

4. **Deterministic**
   
c\_j, a\_ij, and b\_j are all known deterministic values

**Inequalities and equations**

\[ \sum_{j=1}^{m} a_{ij} x_j \leq b_i ; \quad i = 1, 2, \ldots, m \]

The above \( \leq \) constraint can be converted into an equality constraint by introducing a slack variable \( x_{n+i} \).

\[ \Rightarrow \sum_{j=1}^{m} a_{ij} x_j + x_{n+i} = b_i \]

\[ \sum_{j=1}^{m} a_{ij} x_j \geq b_i ; \quad i = 1, 2, \ldots, m \]

The above constraint can be converted into an equality constraint by introducing a surplus variable \( x_{n+i} \Rightarrow ( \text{sometimes denoted by } S_i ) \).

\[ \Rightarrow \sum_{j=1}^{m} a_{ij} x_j - x_{n+i} = b_i \]
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \]

This is not typically done, but has applications in canonical forms and duality theory.

**Nonnegativity restrictions**

The simplex method assumes that the variables introduced in the formulae are all nonnegative.

Q? (i) What if \( x_j \) is *unrestricted* in sign

\[
\Rightarrow x_j = x'_j - x''_j
\]

where \( x'_j \geq 0 \)

\[ + x''_j \geq 0 \]

(ii) \( x_j \geq b_j \) \( \Rightarrow \) lower bound.

\[ \Rightarrow \text{use } x'_j = x_j - b_j \]

\[ + x'_j \geq 0 \]
(iii) \( x_j \leq y_j \), where \( y_j \leq 0 \)

\[ \Rightarrow x_j' = y_j - x_j \]

\[ + x_j' \geq 0 \]

**Relationship between min and max problems**

Max \( Z = \sum_{j=1}^{n} c_j x_j \)

is equivalent to (or \( \equiv \))

Min \(-Z = \sum_{j=1}^{n} -c_j x_j \)

A max(min) problem can be converted into a min(max) problem simply by multiplying the obj func by -1. The optimal obj func value for the original problem is \((-1)\) times the optimal obj func value for the converted problem.

Max \( \sum_{j=1}^{n} c_j x_j = -\text{Min} \sum_{j=1}^{n} -c_j x_j \)

Max \( Z = 10x_1 \)

\[ x_1 \leq 5 \]

\[ x_1 \geq 0 \]

\( Z^* = 50 \)

\(-x_1^* = 5 \)

Min \(-Z = -10x_1 \)

\[ x_1 \leq 5 \]

\[ x_1 \geq 0 \]

Min \( R = 10x_1 \)

\[ x_1 \leq 5 \]

\[ x_1 \geq 0 \]

\( R^* = -50 \)

\(-x_1^* = 5 \)

\( \Rightarrow Z^* = -R^* \)
Standard and canonical forms

A LP problem in standard form consists of constraints that are all equalities and variables that are nonnegative. The simplex method can be applied to a LP problem only after it is converted to a std. form.

**Canonical form — minimization**
- Constraints are all $\geq$, and variables are nonnegative

**Canonical form — maximization**
- Constraints all $\leq$, and variables are nonnegative

Canonical forms are useful in establishing duality relationships.
1. Min \( Z = x_1 - 2x_2 - 3x_3 \)

   St.:
   \[
   \begin{align*}
   x_1 + 2x_2 + x_3 & \leq 14 \\
   x_1 + 2x_2 + 4x_3 & \geq 12 \\
   x_1 - x_2 - x_3 &= 2 \\
   x_1, x_2 & \text{ unrestricted} \\
   x_3 & \leq -3
   \end{align*}
   \]

2. \( x_1 = x_1' - x_1'' \)
   \( x_2 = x_2' - x_2'' \)
   \( x_3' = -3 - x_3 \Rightarrow x_3 = -3 - x_3' \)

   Min \( Z = x_1' - x_1'' - 2x_2' + 2x_2'' + 3x_3' + 9 \)

   St.:
   \[
   \begin{align*}
   x_1' - x_1'' + 2x_2' - 2x_2'' - x_3' & \leq 17 \\
   x_1' - x_1'' + 2x_2' - 2x_2'' - 4x_3' & \geq 24 \\
   x_1' - x_1'' - x_2' + x_2'' - x_3' &= 5 \quad \text{slack} \\
   x_1' - x_1'' + 2x_2' - 2x_2'' - x_3' + x_4' &= 17 \quad \text{surplus} \\
   x_1' - x_1'' + 2x_2' - 2x_2'' - 4x_3' - x_5' &= 24
   \end{align*}
   \]

   \( x_i \geq 0 \quad i = 1, 2, 3 \)
   \( x_i'' \geq 0 \quad j = 1, 2 \)
   \( x_4', x_5' \geq 0 \)
(b) \[ \text{Min } Z = -x_1' - x_1'' - 2x_2' + 2x_2'' + 3x_3' \]

\[ -x_1' + x_1'' - 2x_2' + 2x_2'' + 8x_3' \geq -17 \]

\[ x_1' - x_1'' + 2x_2' - 2x_2'' - 4x_3' \geq 24 \]

\[ x_1' - x_1'' - x_2'' + x_2'' - x_3' \geq 5 \]

\[ -x_1' + x_1'' + x_2'' - x_2'' + x_3' \geq -5 \]

\[ x_2' \geq 0 \quad i = 1,2,3 \]

\[ x_3' \geq 0 \quad j = 1,2 \]

(c) \[ \text{Min } Z = \text{Max } (-2) \]

\[ R = \text{Max } -x_1 + 2x_2 + 3x_3 \]

\[ = \text{Max } -x_1' + x_1'' + 2x_2' - 2x_2'' - 3x_3' \]

\[ \text{st:} \]

1. The same constraints as in the original problem.
2. The “\( \text{Max} \)" as in (b) for standard form.
3. The “\( \text{Min} \)" as in (b) for canonical form.
Max \( Z = x_1 - x_2 \)

s.t.:

1. \(-x_1 + 2x_2 \leq 0\)
2. \(-3x_1 + x_2 \geq -3\)
3. \(x_1, x_2 \geq 0\)

The gradient or the partial derivative vector of the linear function

\[ f_1(x_1, x_2) = -x_1 + 2x_2 \]

\[ \frac{\partial f_1}{\partial x_1} (x_1, x_2) = -1 \]

\[ \frac{\partial f_1}{\partial x_2} (x_1, x_2) = 2 \]

\[ \Rightarrow \text{gradient} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \]

\[ \Rightarrow -x_1 + 2x_2 \text{ increases steeply in the direction} \]

\[ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \] and decreases steeply in the direction

\[ \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \]
\[ f_2(x_1, x_2) = -3x_1 + x_2 \]

The gradient for the linear function
\[ f_2(x_1, x_2) = -3x_1 + x_2 \] is:

\[
\begin{bmatrix}
-3 \\
1
\end{bmatrix}
\]