HW1 posted — Due Thursday, October 4th

2. The feasible region is shaded. It is a **bounded feasible region**.

The number of extreme/corner points is given by:

\[ \binom{m+n}{m} \quad m = \# \text{ of functional constraints} \]
\[ \quad n = \# \text{ of decision variables} \]

\[ = \binom{2+2}{2} \quad \frac{4!}{2! \cdot 2!} = 6 \]

As this is a 2-dimensional problem, the corner point solutions occur at the intersection of two lines.

\[
\begin{align*}
 x_1 &= 0 \quad \text{and} \quad x_2 = 0 \quad \text{intersect at} \quad (0, 0) \\
 x_1 &= 0 \quad \text{and} \quad -x_1 + 2x_2 = 0 \quad \text{at} \quad (0, 0) \\
 x_2 &= 0 \quad \text{and} \quad -x_1 + 2x_2 = 0 \quad \text{at} \quad (0, 0) \\
 -x_1 + 2x_2 = 0 \quad \text{and} \quad -3x_1 + x_2 = -3 \quad \text{at} \quad \left( \frac{3}{2}, \frac{3}{2} \right) \\
 x_2 &= 0 \quad \text{and} \quad -3x_1 + x_2 = -3 \quad \text{at} \quad (1, 0) \\
 x_1 &= 0 \quad \text{and} \quad -3x_1 + x_2 = -3 \quad \text{at} \quad (0, -3)
\end{align*}
\]

The optimal solution is obtained when one of the feasible corner point solutions is given by:

\[ x_1 = 1, \quad x_2 = 0, \quad z_{\text{max}} = 1 \]
(b) \[-x_1 + 2x_2 + x_3 = 0 \rightarrow \text{slack}\]
\[-3x_1 + x_2 - x_4 = -3 \rightarrow \text{surplus}\]

![Graph showing constraints with x3 = 0, x3 + x4 = 0, and x4 = 0.]

(c) Optimal solution is: \((x_1^*, x_2^*) = (1/3, 0)\)

\[x_1^* = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}\]

\[z_1^* = 1\]

Notice that in the above problem, the optimal solution is unique and the solution space is bounded.
The same problem can be modified to demonstrate other types of problems/solutions.

1. Unique opt. soln. in an unbounded region.

Max $Z = x_1 - x_2$

\[ \text{s.t.:} \quad -x_1 + 2x_2 \geq 0 \]
\[ -3x_1 + x_2 \geq -3 \]
\[ x_1, x_2 \geq 0 \]

**Unique opt. soln.** $x^* = \left( \frac{6}{5}, \frac{3}{5} \right)$
(ii) Alternative opt. solns. in a bounded region.

**Challenge:** Make the obj. func. line \( \ell \) to the constraint equation

\[-3x_1 + x_2 = -3\]

slope = 3

\[\Rightarrow \text{change the obj. func. to} \]

\[
\text{Max } Z = 3x_1 - x_2
\]

\[\Rightarrow \text{The complete problem is'}\]

\[
\text{Max } Z = 3x_1 - x_2
\]

s.t.: \( x_1 + 2x_2 \leq 0 \)

\[-3x_1 + x_2 \geq -3\]
Suppose that the obj. func
\[ \text{Max } Z = -3x + x^2 \]

(iii) Alternate opt solution in an unbounded region.

Ans: Intuitively obvious!

\[ \Rightarrow \text{Max } Z = 3x_1 - x_2 \]

s.t.: \[ x_1 + 2x_2 \geq 0 \]
\[ -3x_1 + x_2 \geq -3 \]
\[ x_1, x_2 \geq 0 \]
Unbounded opt. soln. in an unbounded region.

Change the obj. func. to read as:

Max $= -x_1 + x_2$

$s.t.$: $-x_1 + 2x_2 \geq 0$
$-3x_1 + x_2 \geq -3$

The direction $s = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

will make the obj. func. line away from the origin!!

$\Rightarrow$ opt. soln. is unbounded!
(i) Infeasible/empty feasible region.

Max

\[ Z = x_1 - x_2 \]

s.t.: \[ -x_1 + 2x_2 \leq 0 \]
\[ -3x_1 + x_2 \geq -3 \]
\[ x_1 + x_2 \geq 3 \]
\[ x_1, x_2 \geq 0 \]
Requirement Space

The LP problem in std. form is:

\[ \begin{align*}
\text{Min} & \quad 2x \\
\text{s.t.:} & \quad A x = b \\
& \quad x \geq 0 \\
\Rightarrow & \quad \text{Min} \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.:} & \quad \sum_{j=1}^{n} a_j x_j = b \\
& \quad x_j \geq 0 ; \quad j = 1, 2, \ldots, n
\end{align*} \]

The above problem can be restated as:

Given \( q_1, q_2, \ldots, q_n \), find nonnegative (scalars) \( x_1, x_2, \ldots, x_n \) such that

\[ \sum_{j=1}^{n} a_j x_j = b \], and \[ \sum_{j=1}^{n} c_j x_j \text{ is minimum} \].
Ex: Consider 3 vectors \( \vec{a}_1, \vec{a}_2, \text{ and } \vec{a}_3 \) associated with \( x_1, x_2, \text{ and } x_3 \).

\[ \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

The collection of vectors (or the linear combination of vectors) of the form

\[ \sum_{j=1}^{3} \frac{3}{\vec{a}_j} \vec{x}_j \] in the example is a cone generated by \( \vec{a}_1, \vec{a}_2, \vec{a}_3 \).