10-1 THE ADJOINT-NETWORK CONCEPT IN THE FREQUENCY DOMAIN

In Chap. 9, the problems of sensitivity and tolerance analysis were introduced in the time domain for linear memoryless circuits. An efficient technique, utilizing the adjoint-network concept, was described for carrying out the sensitivity calculations. It was also mentioned that the method can readily be extended to the sensitivity analysis of nonlinear memoryless circuits.

It will be shown next that all the results of Chap. 9 are applicable, with trivial modifications, to the important problem of frequency-domain sensitivity analysis of dynamic circuits. As already mentioned in Sec. 2-2, Tellegen's theorem, which is based on the KCL given in (2-28) and on the KVL given in (2-32), remains valid for *any* vector **j** satisfying (2-28) whether it contains physical currents or not and for any vector **v** having a representation given in (2-32) whether it contains physical voltages or not. Since the *complex phasors* associated with the steady-state sine-wave branch currents and voltages of a network certainly satisfy Kirchhoff's laws, the derivations of Sec. 2-2 remain valid. Hence, we have, as in (2-36) and (2-37),

$$V^T\hat{J} = \hat{V}^T J = 0 \qquad (10\text{-}1)$$

Here, the vectors V and J contain the complex phasors of the branch voltages and branch currents, respectively, of the network *N*. $\hat{V}$ and $\hat{J}$ contain the corresponding phasors for another network $\hat{N}$ (the adjoint network) which has the same incidence matrix A as *N*.

From (10-1), we can follow the same steps as in Sec. 9-2 to arrive at the differential Tellegen's theorem valid for the complex phasors of branch voltages and currents:

$$\hat{J}^T\Delta V - \hat{V}^T\Delta J = \sum_{all\ branches} (\hat{J}_k\Delta V_k = \hat{V}_k\Delta J_k) = 0 \qquad (10\text{-}2)$$

which is closely analogous to (9-16).

Next, exactly as for the memoryless circuits, we separate the branches of *N* into branches containing independent voltage sources (for which $\Delta V_k = 0$), branches containing independent current sources (with $\Delta J_k = 0$), and internal branches. The output branch is again treated as a 0-A current source if the output is a voltage and a 0-V voltage source if the output is a current. We choose, as before, all independent sources equal to zero in $\hat{N}$ except for the one which corresponds to the zero-valued source at the output of *N*. This source is set equal to 1 A if the output of *N* is a voltage and to -1V if the output is a current.

With these choices for the generator branches of $\hat{N}$, (10-2) can be rewritten in the form

$$\Delta V_0 \; or \; \Delta J_0 = -\sum_{\substack{all\ internal \\ branches}} (\hat{J}_k \Delta V_k - \hat{V}_k \Delta J_k) \tag{10-3}$$

where V_0 or J_0 is the output of *N*. Assume now that the internal branch phasors V_k and J_k of *N* can be related through a *branch impedance matrix Z*.

$$V_B = ZJ_B \tag{10-4}$$

Here V_B and J_B contain the complex phasors V_k and J_k , respectively, for all internal branches of *N*; Z contains the complex branch impedance and, for CCVS the gains Z_{lm} . Then a derivation, exactly duplicating that which provided (9-35), shows that the branch variables of $\hat{N}$ should be chosen so as to satisfy

$$\hat{V}_B = Z^T \hat{J}_B \tag{10-5}$$

R→ Z
$\Delta v_0 \; or \; \Delta j_0$
$\Delta z = \Delta R$

Hence, the branch impedance matrix of $\hat{N}$ is $\hat{Z} = Z^T$. For this choice of Z, the right-hand side of (10-3) becomes

$$-\hat{J}_B^T \Delta Z J_B = -\sum_{\substack{all\ internal \\ branches}} \hat{J}_l J_m \Delta Z_{lm} = -\sum_{\substack{all \\ impedances}} \hat{J}_l J_l \Delta Z_l - \sum_{\substack{all\ CCVS \\ l \neq m}} \hat{J}_l J_m \Delta Z_{lm} \tag{10-6}$$

Unlike the case in a memoryless circuit, here the ΔZ_l are not simply identical to the element tolerances; they are, however, closely related. From Ohm's law valid for phasors, for an inductive branch

$$\Delta Z_l = j\omega \Delta L_l$$
$$Z_l = j\omega L_l \tag{10-7}$$

$$\Delta Z_l \cong \frac{\partial Z_l}{\partial C_l} \Delta C_l$$

and for a capacitive branch

$$\Delta Z_l = \Delta \frac{1}{j\omega C_l} = -\frac{\Delta C_1}{j\omega C_1^2} \tag{10-8}$$

where the element values of L_l and C_l are treated as toleranced elements. For a resistive branch or for the two branches of a resistive CCVS, $\Delta Z_l \equiv \Delta R_l$ and $\Delta Z_{lm} \equiv \Delta R_{lm}$, as before. Combining these relations with (10-3) and (10-6) gives the sensitivities of the output voltage V_0 :

$$\frac{\partial V_0}{\partial L_l} = -\hat{J}_l J_l j\omega \qquad \frac{\partial V_0}{\partial C_l} = \frac{\hat{J}_l J_l}{j\omega C_1^2}$$
$$\frac{\partial V_0}{\partial R_l} = -\hat{J}_l J_l \qquad \frac{\partial V_0}{\partial R_{lm}} = -\hat{J}_l J_m \tag{10-9}$$

The analogy between (10-9) and Eqs. (9-46) and (9-47) is obvious. For a current output, the output current J_0 replaces V_0 .

Example 10-1 Find the sensitivities of the output voltage V_0 to the element values of the simple circuit shown in Fig. 10-1a.

Since the internal branches of the circuit contain only impedances, $\hat{Z} \equiv Z^T = Z$ and hence the internal branches of *N* and $\hat{N}$ are the same. The only source of $\hat{N}$ is a 1-A (current source) 10-1b.

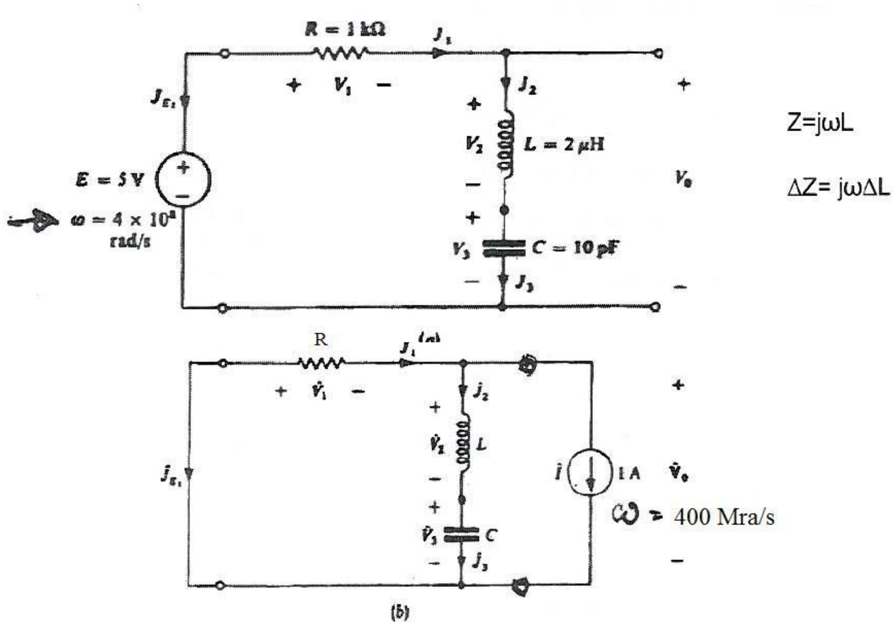


Figure 10-1 (a) Physical circuit; (b) adjoint network in the frequency domain.

From *N*, by inspection,

$$J_1 = J_2 = J_3 = \frac{E}{R + j\omega L + 1/j\omega C} \approx 3.84 - j2.11\ mA$$

For $\hat{N}$, from Fig. 10-1b,

$$\hat{V}_0 = -1\frac{R(j\omega L + 1/j\omega C)}{R + j\omega L + 1/j\omega C} \approx -232 - j422\,V$$

and hence

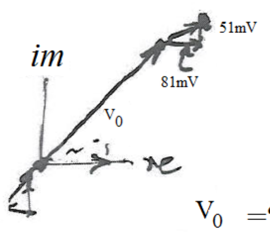
$$\hat{J}_1 = -\frac{\hat{V}_0}{R} \approx +0.232 + j0.422\,A$$

and

$$\hat{J}_2 = \hat{J}_3 = \frac{\hat{V}_0}{j\omega L + 1/j\omega C} \approx -0.767 + j0.422\,A$$

(As a check, we note that $\hat{J}_1 - \hat{J}_2 \cong 1\,A$.) Therefore, using (10-9), we have

$\lvert S_c \rvert = \sqrt{()^2 + ()^2}$ $\quad \Delta c = 1pF \;\rightarrow\; \Delta v \cong 0.081 + j0.0514$



V_0 =?

$$\frac{\partial V_0}{\partial L} = -\hat{J}_2 J_2 j\omega \approx (12.95 + j8.22) \times 10^5\ \ V/H = 1.295 + j0.822\ \ V/\mu H$$

$$S_c = \frac{\partial V_0}{\partial C} = +\frac{\hat{J}_3 J_3}{j\omega C^2} \approx (0.81 + j0.514) \times 10^{11}\,V/F = 0.081 + j0.0514\ \ V/pF$$

and finally

$$\frac{\partial V_0}{\partial R} = -\hat{J}_1 J_1 \approx (-1.78 - j1.131) \times 10^{-3}\ \ V/\Omega = -1.78 - j1.131\ \ V/k\Omega$$

The meaning of the last relation, for example, is of course that a 1-Ω change in R results in a change of –1.78 – j1.131 in the complex output phasor V_0 . (This statement is valid to a first-order approximation only, as discussed in Sec. 9-1.) The other two sensitivities can be interpreted similarly.

From the formula

$$V_0 = E\frac{j\omega L + 1/j\omega C}{R + j\omega L + 1/j\omega C}$$

one can find the sensitivities analytically. For example,

$$\frac{\partial V_0}{\partial L} = E\frac{j\omega R}{(R + j\omega L + 1/j\omega C)^2} \approx (1.295 + j0.822) \times 10^6\ V/H$$

confirming our previously calculated result. The calculation of the other two sensitivies is left to the reader.

It should now be obvious how all other results obtained in Chap. 9 can be extended to frequency-domain sensitivity analysis. If the internal branch relations of *N* can be described by the *branch admittance matrix* Y

$$J_B = YV_B \tag{10-10}$$

Then the adjoint network $\hat{N}$ has the branch admittance matrix $\hat{Y} = Y^T$ and the sensitivities of an output voltage V_0 are given by

$$\begin{aligned} \frac{\partial V_0}{\partial L_l} &= -\frac{\hat{V}_l V_l}{j\omega L_l^2} & \qquad \frac{\partial V_0}{\partial C_l} &= +\hat{V}_l V_l j\omega \\ \frac{\partial V_0}{\partial G_l} &= +\hat{V}_l V_l & \qquad \frac{\partial V_0}{\partial G_{lm}} &= +\hat{V}_l V_m \end{aligned} \tag{10-11}$$

where the last relation gives the sensitivity to the gain of a VCCS.

For the calculation of the sensitivities of an output *current* J_0 in all relations J_0 replaces V_0 .

The analogy between (10-11) on the one hand and Eqs. (9-51) and (9-52) on the other is manifest.

Often only a hybrid formulation of the internal branch relations is possible, i.e., often

$$\begin{bmatrix} J_{B_1} \\ V_{B_2} \end{bmatrix} = \begin{bmatrix} Y & A \\ M & Z \end{bmatrix} \begin{bmatrix} V_{B_1} \\ J_{B_2} \end{bmatrix} \tag{10-12}$$

represents the only feasible description of the equations connecting the branch vectors

$$J_B = \begin{bmatrix} J_{B_1} \\ J_{B_2} \end{bmatrix} \quad \text{and} \quad V_B = \begin{bmatrix} V_{B_1} \\ V_{B_2} \end{bmatrix} \tag{10-13}$$

Then a derivation essentially identical to that leading to (9-70) to (9-74) shows that the internal structure of $\hat{N}$ must be such that

$$\begin{bmatrix} \hat{J}_{B_1} \\ \hat{V}_{B_2} \end{bmatrix} = \begin{bmatrix} Y^T & -M^T \\ -A^T & Z^T \end{bmatrix} \begin{bmatrix} \hat{V}_{B_1} \\ \hat{J}_{B_2} \end{bmatrix} \tag{10-14}$$

holds. Then the sensitivities of the output voltage V_0 are given by the relations

$$\frac{\partial V_0}{\partial H_{lm}} = \begin{cases} \hat{V}_l V_m & if\ H_{lm}\ is\ in\ Y \\ \hat{V}_l J_m & if\ H_{lm}\ is\ in\ A \\ -\hat{J}_l V_m & if\ H_{lm}\ is\ in\ M \\ -\hat{J}_l J_m & if\ H_{lm}\ is\ in\ Z \end{cases} \tag{10-15}$$

V_0 is to be replaced by J_0 if the output is the current J_0 .

Note that the elements of A are the CCCS gains and hence are usually real. Similarly, the elements of M are the (normally real) VCVS gains. The diagonal elements of Y are the branch admittances and are therefore in the form G_l , $j\omega C_l$, or $1/j\omega L$; the off-diagonal elements are the (usually real) gains of the VCCSs. Dually, the diagonal elements of Z are branch impedances (R_l, $j\omega L_l$, or $1/j\omega C_l$), while its off-diagonal elements are the (normally real) CCVS gains. For example, if $Z_l = j\omega L_l$ is the H_{ll} element which is contained in Z, then, by (10-15), $\partial V_0/\partial L_1 = (\partial V_0/\partial H_{ll})(\partial H_{LL}/\partial L_l) = \left(-\hat{J}_l J_l\right)(j\omega)$. Other sensitivities can be found similarly.

Also, group delay $\partial\phi/\partial\omega$, pole & zero sensitivities, etc. can be found via $\widehat{N}$

10-2 Use the adjoint-network concept to calculate the sensitivities of the output V_0 of the active integrator circuit shown in Fig. 10-14. Check your results using analytic differentiation.

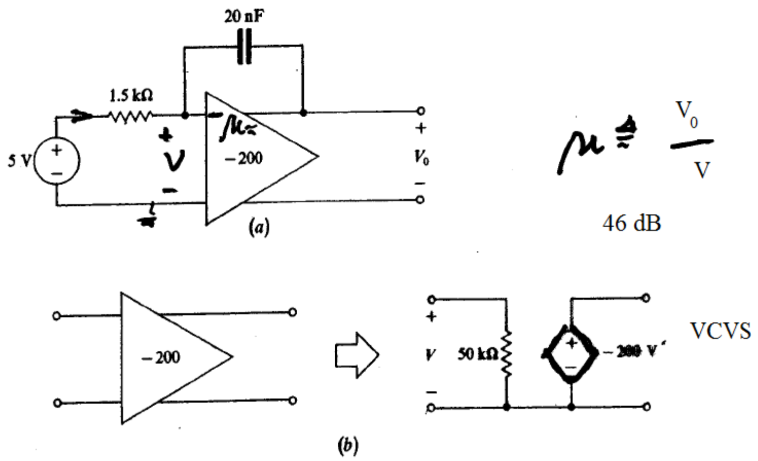
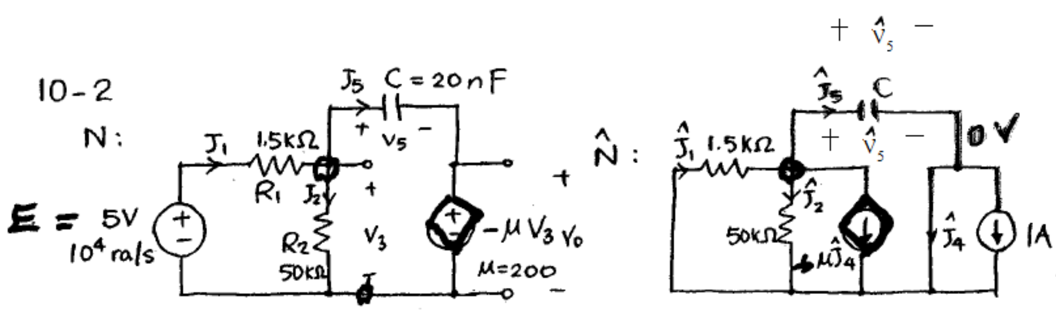


Figure 10-14 (a) Active integrator; (b) amplifier model. The radian frequency is $\boldsymbol{\omega} = \mathbf{10^4}$rad/s.



$$N: \frac{V_3 - E}{R} + \frac{V_3}{R_2} + j\omega C(1+\mu)V_3 = 0$$

$$V_3 = \frac{E}{1 + \frac{R_1}{R_2} + j\omega R_1 C(1+\mu)} = 1.4159 \times 10^{-3} - j8.2895 \times 10^{-2} \quad V$$

$$J_1 = \frac{E - V_3}{R_1} = 3.3324 \times 10^{-3} + j5.5263 \times 10^{-5} \quad A$$

$$J_2 = \frac{V_3}{R_2} = 2.5319 \times 10^{-8} - j1.6579 \times 10^{-6} \quad A$$

$$V_5 = (1+\mu)V_3 = 0.28461 - j16.662 \quad V$$

$\hat{N}$: $\hat{J}_4 = \hat{J}_5 - 1 = sC\hat{V}_5 - 1$

$$\frac{\hat{V}_5}{R_1} + \frac{\hat{V}_5}{R_2} + \mu\left(sC\hat{V}_5 - 1\right) + sC\hat{V}_5 = 0$$

$$\therefore \hat{V}_5 = \frac{\mu}{\frac{1}{R_1} + \frac{1}{R_2} + (1+\mu)sC} = 8.4957 \times 10^1 - j4.9737 \times 10^3$$

$$\hat{J}_1 = -\frac{\hat{V}_5}{R_1} = -5.6638 \times 10^{-2} + j3.3158$$

$$\hat{J}_2 = \frac{\hat{V}_5}{R_2} = 1.6994 \times 10^{-3} - j9.9473 \times 10^{-2}$$

$$\hat{J}_4 = sC\hat{V}_5 - 1 = -0.99667 + j5.6921 \times 10^{-5}$$

$$\frac{\partial V_0}{\partial C} = j\omega\hat{V}_5 V_5 = 2.8311 \times 10^7 - j8.2846 \times 10^8 \quad V/F$$

$$\frac{\partial V_0}{\partial R_1} = -\hat{J}_1 J_1 = 3.7198 \times 10^{-4} - j1.1046 \times 10^{-2} \quad V/\Omega$$

$$\frac{\partial V_0}{\partial R_2} = -\hat{J}_2 J_2 = 1.6487 \times 10^{-7} + j5.634 \times 10^{-9} \quad V/\Omega$$

$$\frac{\partial V_0}{\partial \mu} = V_3\hat{J}_4 = -1.4065 \times 10^{-3} + j8.2618 \times 10^{-2}$$