Final Exam 2018

ECE 351

1) a) <u>linear</u>: $ay_1(t) = a\int_{t-2}^{t+1} (t-\tau)^2 x_1(\tau)\,d\tau$

$$by_2(t) = b\int_{t-2}^{t+1} (t-\tau)^2 x_2(\tau)\,d\tau$$

$$H(ax_1+bx_2) = \int_{t-2}^{t+1} (t-\tau)^2 (a x_1(\tau) + b x_2(\tau))\,d\tau = \int_{t-2}^{t+1} (a(t-\tau)^2 x_1(\tau) + b(t-\tau)^2 x_2(\tau))\,d\tau$$

$$= a\int_{t-2}^{t+1} (t-\tau)^2 x_1(\tau)\,d\tau + b\int_{t-2}^{t+1} (t-\tau)^2 x_2(\tau)\,d\tau = ay_1(t) + by_2(t)$$

<u>Time invariant</u>:

$$y(t-t_0) = \int_{(t-t_0)-2}^{(t-t_0+1)} ((t-t_0)-\tau)^2 x(\tau)\,d\tau$$

$$H(x(t-t_0)) = \int_{t-2}^{t+1} (t-\tau)^2 x(\tau-t_0)\,d\tau$$

let $\tau - t_0 = t'$, $d\tau = dt'$ and $\tau = t'+t_0$

$$H(x(t-t_0)) = \int_{t-t_0-2}^{t-t_0+1} (t-t'-t_0)^2 x(t')\,dt'$$

$$= \int_{(t-t_0)-2}^{(t-t_0)+1} ((t-t_0)-t')^2 x(t')\,dt' = y(t-t_0)$$

b) $h(t) = \int_{t-2}^{t+1} (t-\tau)^2 \delta(\tau)\,d\tau$

- If $\begin{cases} t+1 \geq 0 \\ t-2 \leq 0 \end{cases} \Leftrightarrow t \in [-1;2]$

- Then $h(t) = \int_{t-2}^{t+1} (t-\tau)^2 \delta(\tau)\,d\tau = (t-0)^2\delta(0) = t^2$

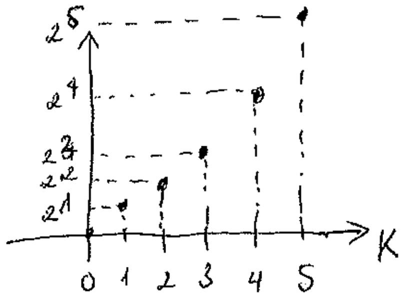
So $h(t) = t^2\,[u(t+1) - u(t-2)]$

c) • $\int_{-1}^{2} |h(t)|^2\,dt = \int_{-1}^{2} t^2\,dt$ = finite $< +\infty$ since this is a finite integral $\Rightarrow$ BIBO!

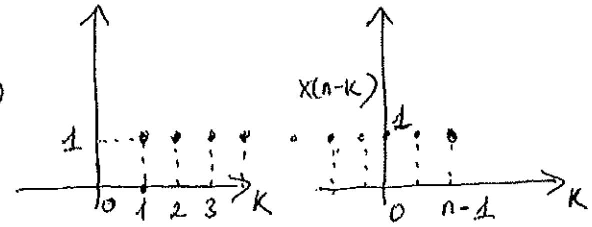
× $h(-0.5) = \frac{1}{4} > 0 \Rightarrow$ Not casual!

2)

a) $h(k)$



$X(K)$



b) $y(n) = x(n) * h(n)$

* $n-1 < 0 \Leftrightarrow n < 1 \Rightarrow y(n) = 0$

* $\begin{cases} n-1 \geq 0 \\ n-1 < 6 \end{cases} \Leftrightarrow n \in [1,7) \Rightarrow y(n) = \sum_{k=0}^{n-1} 2^k = 2^n - 1$

* $n-1 \geq 6 \Leftrightarrow n \geq 7 \Rightarrow y(n) = \sum_{k=0}^{5} 2^k = 2^6 - 1$

3)

a) $\left.\begin{array}{l} \cos^2\left(\frac{2\pi n}{3}\right) \Rightarrow \omega_1 = \left(\frac{2\pi}{3}\right).2 = \frac{4\pi}{3} \Rightarrow N_1 = 3 \\ e^{j\pi n/2} \Rightarrow \omega_2 = \frac{\pi}{2} \Rightarrow N_2 = 4 \end{array}\right\} N = 12 \Rightarrow \Omega = \frac{2\pi}{12} = \frac{\pi}{6}$

$$X(n) = \left(\frac{e^{j2\pi n/3} + e^{-j2\pi n/3}}{2}\right)^2 + e^{j\pi n/2} = \frac{1}{4} e^{j4\pi n/3} + \frac{1}{4} e^{-j4\pi n/3} + \frac{1}{2} + e^{j\pi n/2}$$

$$= \frac{1}{4} e^{j8\frac{\pi}{6}n} + \frac{1}{4} e^{j(-8)\frac{\pi}{6}n} + 1.\, e^{j3\frac{\pi}{6}n} + \frac{1}{2} e^{j.0.n}$$

$\Rightarrow X(k=8) = \frac{1}{4}$, $X(k=-8) = \frac{1}{4}$, $X(k=3) = 1$, $X(k=0) = 1/2$

b) $$X(j\omega) = \int_{-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} 2^{-|t|} \delta(t-i) e^{-j\omega t} dt = \sum_{i=-\infty}^{+\infty} \int_{-\infty}^{+\infty} 2^{-|t|} \delta(t-i) e^{-j\omega t} dt$$

$$= \sum_{i=-\infty}^{+\infty} 2^{-|i|} e^{-j\omega i} = \sum_{i=-\infty}^{0} 2^{i} e^{-j\omega i} + \sum_{i=0}^{+\infty} e^{-i\omega j} 2^{-i} - 1$$

$(k=-i)$ $$= \sum_{k=0}^{+\infty} \left(\frac{e^{j\omega}}{2}\right)^k + \sum_{i=0}^{+\infty} \left(\frac{1}{2e^{j\omega}}\right)^i - 1 = \frac{1}{1-(2e^{-j\omega})^{-1}} + \frac{1}{1-(2e^{j\omega})^{-1}} - 1$$

4) * $e^{-at}u(t) \rightarrow \frac{1}{a+j\omega}$, thus, $e^{-\frac{1}{2}t}u(t) \rightarrow \frac{1}{\frac{1}{2}+j\omega}$

Thus $e^{-t/2}u(t-2) = \frac{1}{e}\, e^{-\frac{1}{2}(t-2)}u(t-2) \longrightarrow \frac{1}{e}\cdot e^{-j\omega 2}\cdot\frac{1}{\frac{1}{2}+j\omega}$ ①

(time shift property)

* $\cos(\pi t)\, e^{-|t|} = \frac{1}{2}\, e^{j\pi t} e^{-|t|} + \frac{1}{2}\, e^{-j\pi t} e^{-|t|}$

$e^{-a|t|} \rightarrow \frac{2a}{a^2+\omega^2}$, thus, $e^{-|t|} = \frac{2}{1+\omega^2}$

Next, using frequency Shift:

$\frac{1}{2} e^{j\pi t} e^{-|t|} \longrightarrow \frac{1}{2}\,\frac{2}{1+(\omega-\pi)^2} = \frac{1}{(\omega-\pi)^2+1}$

$\frac{1}{2} e^{-j\pi t} e^{-|t|} \longrightarrow \frac{1}{2}\,\frac{2}{1+(\omega+\pi)^2} = \frac{1}{(\omega+\pi)^2+1}$

So $\cos(\pi t)\, e^{-|t|} \rightarrow \frac{1}{(\omega-\pi)^2+1} + \frac{1}{(\omega+\pi)^2+1}$

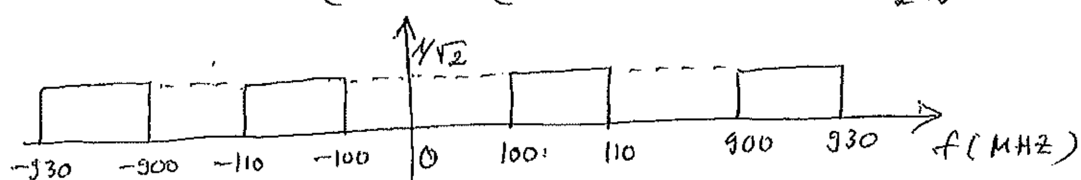
Finally $\frac{d\left(\cos(\pi t)\, e^{-|t|}\right)}{dt} \rightarrow j\omega\left(\frac{1}{(\omega-\pi)^2+1} + \frac{1}{(\omega+\pi)^2+1}\right)$ ②

* Combine ① and ②

$$X(t) = j\omega\left(\frac{1}{(\omega-\pi)^2+1} + \frac{1}{(\omega+\pi)^2+1}\right)\cdot\frac{1}{e}\, e^{-j2\omega}\,\frac{1}{\frac{1}{2}+j\omega}$$

5)

a)



Signal is casted to positive and negative frequency. We want power = 1

So each part only has amplitude $\frac{1}{\sqrt{2}}$ since $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$.

b) $H_1(j\omega) = \frac{1}{\sqrt{2}}\left(S_W(j(\omega-105)) + S_W(j(\omega+105))\right)$, $W_1 = 5$

$H_2(j\omega) = \frac{1}{\sqrt{2}}\left(S_W(j(\omega+915)) + S_W(j(\omega-915))\right)$, $W_2 = 15$

d) Shifted in frequency property:

$h_1(t) = \frac{1}{\sqrt{2}}\, e^{j\cdot 105\cdot t}\, s_W(t) + \frac{1}{\sqrt{2}}\, e^{j(-105)t}\, s_W(t)$

$h_2(t) = \frac{1}{\sqrt{2}}\, e^{j\,915\cdot t}\, s_W(t) + \frac{1}{\sqrt{2}}\, e^{j(-915)t}\, s_W(t)$