MIDTERM 2

Solution

1) $\sin\left(\frac{\pi k}{2}\right) = \dfrac{e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}}{2j}$ $\quad \cos\left(\frac{\pi k}{3}\right) = \dfrac{e^{j\frac{\pi k}{3}} + e^{-j\frac{\pi k}{3}}}{2}$

$$\Rightarrow X[k] = \frac{1 - e^{-j\pi k}}{2j} + \frac{e^{j\frac{\pi k}{3}} + e^{-j\frac{\pi k}{3}}}{2} + 1$$

$$\left.\begin{array}{l} N_1 = \frac{2\pi}{\pi} = 2 \\ N_2 = \frac{2\pi}{\pi/3} = 6 \end{array}\right\} \Rightarrow N = 6 \Rightarrow \Omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$X[k] = \left(\frac{1}{2j} + 1\right) + \frac{-1}{2j} e^{-j\pi k} + \frac{1}{2} e^{j\frac{\pi k}{3}} + \frac{1}{2} e^{-j\frac{\pi k}{3}}$$

$$= \frac{1}{N}\left(6\left(\frac{1}{2j}+1\right)\right) + \frac{1}{N}\left(-\frac{3}{j}\right)e^{+j(-3)\cdot\frac{\pi}{3}k} + \frac{1}{N}\cdot 3 \cdot e^{j\cdot 1\cdot\frac{\pi}{3}k} + \frac{1}{N} 3 e^{j(-1)\frac{\pi}{3}k}$$

$\Rightarrow X(0) = 6\left(\frac{1}{2j}+1\right)$

$X(-3) = \frac{-3}{j} = 3j$

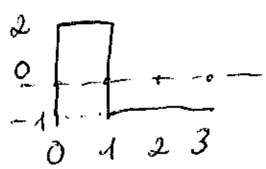
$X(1) = 3$

$X(-1) = 3$

O.W $(X(n) = 0)$ $\forall n \in [-2; 3]$; $n \neq 0, -3, 1, -1$

2) $T = 3$

$\omega_0 = \frac{2\pi}{3}$



$$X(k) = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{3}\int_0^3 x(t) e^{-jk\frac{2\pi}{3}t}\, dt$$

$$= \frac{1}{3}\int_0^1 2 e^{-jk\frac{2\pi}{3}t}\, dt - \frac{1}{3}\int_1^3 e^{-j\frac{2\pi}{3}kt}\, dt$$

$$= \frac{2}{3}\left(-\frac{e^{-jk\frac{2\pi}{3}t}}{jk\frac{2\pi}{3}}\right)\Bigg|_0^1 - \frac{1}{3}\left(-\frac{e^{-j\frac{2\pi}{3}kt}}{j\frac{2\pi}{3}k}\right)\Bigg|_1^3$$

$$= \frac{1 - e^{-j\frac{2\pi}{3}k}}{jk\pi} + \frac{e^{-j2\pi k} - e^{-j\frac{2\pi}{3}k}}{2j\pi k}$$

$e^{-j2\pi K} \rightarrow 1$ since K is an integer. Thus,

$$X(k) = \frac{3}{2}\left(\frac{1 - e^{\frac{-j2\pi}{3}K}}{j\pi K}\right)$$

3) a) $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau = \int_0^{+\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau = \int_0^{+\infty} e^{-(1+j\omega)\tau}\, d\tau$

$$= \left.\frac{e^{-(1+j\omega)\tau}}{-(1+j\omega)}\right|_0^{+\infty} = 0 - \frac{1}{-(1+j\omega)} = \frac{1}{1+j\omega} \quad (1)$$

b) $x(t) = \cos^2\left(\frac{\pi t}{2}\right) + 1 = \left(\frac{e^{\frac{j\pi t}{2}} + e^{-j\frac{\pi t}{2}}}{2}\right)^2 + 1$

$$= \frac{e^{j\pi t} + e^{-j\pi t} + 2}{4} + 1$$

$$= \frac{3}{2} + \frac{1}{4} e^{j\pi t} + \frac{1}{4} e^{-j\pi t}$$

$$\left(c_0 = \frac{3}{2},\ \omega_0 = 0\right) \quad \left(c_1 = \frac{1}{4}\ \omega_1 = \pi\right) \quad \left(c_2 = \frac{1}{4},\ \omega_2 = -\pi\right)$$

$$\Rightarrow y(t) = \frac{3}{2}H(\omega_0) + \frac{1}{4} e^{j\pi t} H(\omega_1) + \frac{1}{4} e^{-j\pi t} H(\omega_2)$$

$$= \frac{3}{2} + \frac{1}{4}\,\frac{1}{1+j\pi}\, e^{j\pi t} + \frac{1}{4} e^{-j\pi t}\,\frac{1}{1-j\pi}$$

c) DC, for example put $x(t) = a = a e^{j0.t} \Rightarrow \omega = 0$

plug in (1) $\Rightarrow H(j\omega) = 1$, similar to part 3-b

$$y(t)\Big|_{x(t)=a} = a \cdot H(\omega = 0) = a.1 = a \Rightarrow \text{No change!}$$