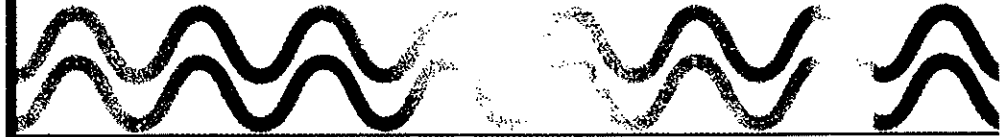


C

Tables of Fourier Representations and Properties



C.1 Basic Discrete-Time Fourier Series Pairs

Time Domain	Frequency Domain
$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk n \Omega_0}$ <p style="text-align: center;">Period = N</p>	$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk n \Omega_0}$ $\Omega_0 = \frac{2\pi}{N}$
$x[n] = \begin{cases} 1, & n \leq M \\ 0, & M < n \leq N/2 \end{cases}$ $x[n] = x[n + N]$	$X[k] = \frac{\sin\left(k \frac{\Omega_0}{2} (2M + 1)\right)}{N \sin\left(k \frac{\Omega_0}{2}\right)}$
$x[n] = e^{jp \Omega_0 n}$	$X[k] = \begin{cases} 1, & k = p, p \pm N, p \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \cos(p \Omega_0 n)$	$X[k] = \begin{cases} \frac{1}{2}, & k = \pm p, \pm p \pm N, \pm p \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \sin(p \Omega_0 n)$	$X[k] = \begin{cases} \frac{1}{2j}, & k = p, p \pm N, p \pm 2N, \dots \\ \frac{-1}{2j}, & k = -p, -p \pm N, -p \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = 1$	$X[k] = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \sum_{p=-\infty}^{\infty} \delta[n - pN]$	$X[k] = \frac{1}{N}$

C.2 Basic Fourier Series Pairs

Time Domain	Frequency Domain
$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$ <p style="text-align: center;">Period = T</p>	$X[k] = \frac{1}{T} \int_0^T x(t)e^{-jk\omega_0 t} dt$ $\omega_0 = \frac{2\pi}{T}$
$x(t) = \begin{cases} 1, & t \leq T_0 \\ 0, & T_0 < t \leq T/2 \end{cases}$	$X[k] = \frac{\sin(k\omega_0 T_0)}{k\pi}$
$x(t) = e^{jp\omega_0 t}$	$X[k] = \delta[k - p]$
$x(t) = \cos(p\omega_0 t)$	$X[k] = \frac{1}{2}\delta[k - p] + \frac{1}{2}\delta[k + p]$
$x(t) = \sin(p\omega_0 t)$	$X[k] = \frac{1}{2j}\delta[k - p] - \frac{1}{2j}\delta[k + p]$
$x(t) = \sum_{p=-\infty}^{\infty} \delta(t - pT)$	$X[k] = \frac{1}{T}$

C.3 Basic Discrete-Time Fourier Transform Pairs

Time Domain	Frequency Domain
$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega$	$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$
$x[n] = \begin{cases} 1, & n \leq M \\ 0, & \text{otherwise} \end{cases}$	$X(e^{j\Omega}) = \frac{\sin\left[\Omega\left(\frac{2M+1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$
$x[n] = \alpha^n u[n], \quad \alpha < 1$	$X(e^{j\Omega}) = \frac{1}{1 - \alpha e^{-j\Omega}}$
$x[n] = \delta[n]$	$X(e^{j\Omega}) = 1$
$x[n] = u[n]$	$X(e^{j\Omega}) = \frac{1}{1 - e^{-j\Omega}} + \pi \sum_{p=-\infty}^{\infty} \delta(\Omega - 2\pi p)$
$x[n] = \frac{1}{\pi n} \sin(Wn), \quad 0 < W \leq \pi$	$X(e^{j\Omega}) = \begin{cases} 1, & \Omega \leq W \\ 0, & W < \Omega \leq \pi \end{cases} \quad X(e^{j\Omega}) \text{ is } 2\pi \text{ periodic}$
$x[n] = (n+1)\alpha^n u[n]$	$X(e^{j\Omega}) = \frac{1}{(1 - \alpha e^{-j\Omega})^2}$

C.4 Basic Fourier Transform Pairs

Time Domain	Frequency Domain
$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
$x(t) = \begin{cases} 1, & t \leq T_0 \\ 0, & \text{otherwise} \end{cases}$	$X(j\omega) = \frac{2 \sin(\omega T_0)}{\omega}$
$x(t) = \frac{1}{\pi t} \sin(Wt)$	$X(j\omega) = \begin{cases} 1, & \omega \leq W \\ 0, & \text{otherwise} \end{cases}$
$x(t) = \delta(t)$	$X(j\omega) = 1$
$x(t) = 1$	$X(j\omega) = 2\pi\delta(\omega)$
$x(t) = u(t)$	$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$
$x(t) = e^{-at}u(t), \quad \text{Re}\{a\} > 0$	$X(j\omega) = \frac{1}{a + j\omega}$
$x(t) = te^{-at}u(t), \quad \text{Re}\{a\} > 0$	$X(j\omega) = \frac{1}{(a + j\omega)^2}$
$x(t) = e^{-a t }, \quad a > 0$	$X(j\omega) = \frac{2a}{a^2 + \omega^2}$
$x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$	$X(j\omega) = e^{-\omega^2/2}$

C.5 Fourier Transform Pairs for Periodic Signals

Periodic Time-Domain Signal	Fourier Transform
$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$	$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_0)$
$x(t) = \cos(\omega_0 t)$	$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$x(t) = \sin(\omega_0 t)$	$X(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$
$x(t) = e^{j\omega_0 t}$	$X(j\omega) = 2\pi\delta(\omega - \omega_0)$
$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$	$X(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$
$x(t) = \begin{cases} 1, & t \leq T_0 \\ 0, & T_0 < t < T/2 \end{cases}$ $x(t + T) = x(t)$	$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2 \sin(k\omega_0 T_0)}{k} \delta(\omega - k\omega_0)$

C.6 Discrete-Time Fourier Transform Pairs for Periodic Signals

<i>Periodic Time-Domain Signal</i>	<i>Discrete-Time Fourier Transform</i>
$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$	$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_0)$
$x[n] = \cos(\Omega_1 n)$	$X(e^{j\Omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi) + \delta(\Omega + \Omega_1 - k2\pi)$
$x[n] = \sin(\Omega_1 n)$	$X(e^{j\Omega}) = \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi) - \delta(\Omega + \Omega_1 - k2\pi)$
$x[n] = e^{j\Omega_1 n}$	$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_1 - k2\pi)$
$x[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN)$	$X(e^{j\Omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{k2\pi}{N}\right)$

C.7 Properties of Fourier Representations

Property	<p style="text-align: center;"><i>Fourier Transform</i></p> $x(t) \xleftrightarrow{FT} X(j\omega)$ $y(t) \xleftrightarrow{FT} Y(j\omega)$	<p style="text-align: center;"><i>Fourier Series</i></p> $x(t) \xleftrightarrow{FS; \omega_0} X[k]$ $y(t) \xleftrightarrow{FS; \omega_0} Y[k]$ <p style="text-align: center;">Period = T</p>
Linearity	$ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$	$ax(t) + by(t) \xleftrightarrow{FS; \omega_0} aX[k] + bY[k]$
Time shift	$x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$	$x(t - t_0) \xleftrightarrow{FS; \omega_0} e^{-jk\omega_0 t_0} X[k]$
Frequency shift	$e^{j\gamma t} x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$	$e^{jk_0 t} x(t) \xleftrightarrow{FS; \omega_0} X[k - k_0]$
Scaling	$x(at) \xleftrightarrow{FT} \frac{1}{ a } X\left(\frac{j\omega}{a}\right)$	$x(at) \xleftrightarrow{FS; a\omega_0} X[k]$
Differentiation in time	$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(j\omega)$	$\frac{d}{dt} x(t) \xleftrightarrow{FS; \omega_0} jk\omega_0 X[k]$
Differentiation in frequency	$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$	<p style="text-align: center;">—</p>
Integration/Summation	$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$	<p style="text-align: center;">—</p>
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t - \tau) d\tau \xleftrightarrow{FT} X(j\omega)Y(j\omega)$	$\int_0^T x(\tau)y(t - \tau) d\tau \xleftrightarrow{FS; \omega_0} TX[k]Y[k]$
Multiplication	$x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\nu)Y(j(\omega - \nu)) d\nu$	$x(t)y(t) \xleftrightarrow{FS; \omega_0} \sum_{l=-\infty}^{\infty} X[l]Y[k - l]$
Parseval's Theorem	$\int_{-\infty}^{\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) ^2 d\omega$	$\frac{1}{T} \int_0^T x(t) ^2 dt = \sum_{k=-\infty}^{\infty} X[k] ^2$
Duality	$X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$ $X(e^{j\Omega}) \xleftrightarrow{FS; 1} x[-k]$
Symmetry	$x(t) \text{ real} \xleftrightarrow{FT} X^*(j\omega) = X(-j\omega)$ $x(t) \text{ imaginary} \xleftrightarrow{FT} X^*(j\omega) = -X(-j\omega)$ $x(t) \text{ real and even} \xleftrightarrow{FT} \text{Im}\{X(j\omega)\} = 0$ $x(t) \text{ real and odd} \xleftrightarrow{FT} \text{Re}\{X(j\omega)\} = 0$	$x(t) \text{ real} \xleftrightarrow{FS; \omega_0} X^*[k] = X[-k]$ $x(t) \text{ imaginary} \xleftrightarrow{FS; \omega_0} X^*[k] = -X[-k]$ $x(t) \text{ real and even} \xleftrightarrow{FS; \omega_0} \text{Im}\{X[k]\} = 0$ $x(t) \text{ real and odd} \xleftrightarrow{FS; \omega_0} \text{Re}\{X[k]\} = 0$

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C.7 (continued)

Property	<p style="text-align: center;">Discrete-Time FT</p> $x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$ $y[n] \xleftrightarrow{\text{DTFT}} Y(e^{j\Omega})$	<p style="text-align: center;">Discrete-Time FS</p> $x[n] \xleftrightarrow{\text{DTFS}; \Omega_0} X[k]$ $y[n] \xleftrightarrow{\text{DTFS}; \Omega_0} Y[k]$ <p style="text-align: center;">Period = N</p>
Linearity	$ax[n] + by[n] \xleftrightarrow{\text{DTFT}} aX(e^{j\Omega}) + bY(e^{j\Omega})$	$ax[n] + by[n] \xleftrightarrow{\text{DTFS}; \Omega_0} aX[k] + bY[k]$
Time shift	$x[n - n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(e^{j\Omega})$	$x[n - n_0] \xleftrightarrow{\text{DTFS}; \Omega_0} e^{-jk\Omega_0 n_0} X[k]$
Frequency shift	$e^{j\Gamma n} x[n] \xleftrightarrow{\text{DTFT}} X(e^{j(\Omega - \Gamma)})$	$e^{jk_0 \Omega_0 n} x[n] \xleftrightarrow{\text{DTFS}; \Omega_0} X[k - k_0]$
Scaling	$x_z[n] = 0, \quad n \neq 0, \pm p, \pm 2p, \pm 3p, \dots$ $x_z[pn] \xleftrightarrow{\text{DTFT}} X_z(e^{j\Omega/p})$	$x_z[n] = 0, \quad n \neq 0, \pm p, \pm 2p, \pm 3p, \dots$ $x_z[pn] \xleftrightarrow{\text{DTFS}; p\Omega_0} pX_z[k]$
Differentiation in time	—	—
Differentiation in frequency	$-jnx[n] \xleftrightarrow{\text{DTFT}} \frac{d}{d\Omega} X(e^{j\Omega})$	—
Integration/Summation	$\sum_{k=-\infty}^n x[k] \xleftrightarrow{\text{DTFT}} \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}}$ $+ \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$	—
Convolution	$\sum_{l=-\infty}^{\infty} x[l]y[n-l] \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})Y(e^{j\Omega})$	$\sum_{l=0}^{N-1} x[l]y[n-l] \xleftrightarrow{\text{DTFS}; \Omega_0} NX[k]Y[k]$
Multiplication	$x[n]y[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Gamma})Y(e^{j(\Omega - \Gamma)}) d\Gamma$	$x[n]y[n] \xleftrightarrow{\text{DTFS}; \Omega_0} \sum_{l=0}^{N-1} X[l]Y[k-l]$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} x[n] ^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) ^2 d\Omega$	$\frac{1}{N} \sum_{n=0}^{N-1} x[n] ^2 = \sum_{k=0}^{N-1} X[k] ^2$
Duality	$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\Omega})$ $X(e^{j\Omega}) \xleftrightarrow{\text{FS}; 1} x[-k]$	$X[n] \xleftrightarrow{\text{DTFS}; \Omega_0} \frac{1}{N} x[-k]$
Symmetry	$x[n] \text{ real} \xleftrightarrow{\text{DTFT}} X^*(e^{j\Omega}) = X(e^{-j\Omega})$ $x[n] \text{ imaginary} \xleftrightarrow{\text{DTFT}} X^*(e^{j\Omega}) = -X(e^{-j\Omega})$ $x[n] \text{ real and even} \xleftrightarrow{\text{DTFT}} \text{Im}\{X(e^{j\Omega})\} = 0$ $x[n] \text{ real and odd} \xleftrightarrow{\text{DTFT}} \text{Re}\{X(e^{j\Omega})\} = 0$	$x[n] \text{ real} \xleftrightarrow{\text{DTFS}; \Omega_0} X^*[k] = X[-k]$ $x[n] \text{ imaginary} \xleftrightarrow{\text{DTFS}; \Omega_0} X^*[k] = -X[-k]$ $x[n] \text{ real and even} \xleftrightarrow{\text{DTFS}; \Omega_0} \text{Im}\{X[k]\} = 0$ $x[n] \text{ real and odd} \xleftrightarrow{\text{DTFS}; \Omega_0} \text{Re}\{X[k]\} = 0$

C.8 Relating the Four Fourier Representations

Let

$$\begin{aligned} g(t) &\xleftrightarrow{\text{FS}; \omega_o = 2\pi/T} G[k] \\ v[n] &\xleftrightarrow{\text{DTFT}} V(e^{j\Omega}) \\ w[n] &\xleftrightarrow{\text{DTFS}; \Omega_o = 2\pi/N} W[k] \end{aligned}$$

■ C.8.1 FT REPRESENTATION FOR A CONTINUOUS-TIME PERIODIC SIGNAL

$$g(t) \xleftrightarrow{\text{FT}} G(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} G[k] \delta(\omega - k\omega_o)$$

■ C.8.2 DTFT REPRESENTATION FOR A DISCRETE-TIME PERIODIC SIGNAL

$$w[n] \xleftrightarrow{\text{DTFT}} W(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} W[k] \delta(\Omega - k\Omega_o)$$

■ C.8.3 FT REPRESENTATION FOR A DISCRETE-TIME NONPERIODIC SIGNAL

$$v_s(t) = \sum_{n=-\infty}^{\infty} v[n] \delta(t - nT_s) \xleftrightarrow{\text{FT}} V_s(j\omega) = V(e^{j\Omega}) \Big|_{\Omega=\omega T_s}$$

■ C.8.4 FT REPRESENTATION FOR A DISCRETE-TIME NONPERIODIC SIGNAL

$$w_s(t) = \sum_{n=-\infty}^{\infty} w[n] \delta(t - nT_s) \xleftrightarrow{\text{FT}} W_s(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} W[k] \delta\left(\omega - \frac{k\Omega_o}{T_s}\right)$$

C.9 Sampling and Aliasing Relationships

Let

$$\begin{aligned} x(t) &\xleftrightarrow{\text{FT}} X(j\omega) \\ v[n] &\xleftrightarrow{\text{DTFT}} V(e^{j\Omega}) \end{aligned}$$

■ C.9.1 IMPULSE SAMPLING FOR CONTINUOUS-TIME SIGNALS

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) \xleftrightarrow{\text{FT}} X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\frac{2\pi}{T_s}\right)\right)$$

Sampling interval T_s , $X_s(j\omega)$ is $2\pi/T_s$ periodic.

■ C.9.2 SAMPLING A DISCRETE-TIME SIGNAL

$$y[n] = v[qn] \xleftrightarrow{\text{DTFT}} Y(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-1} V(e^{j(\Omega - m2\pi)/q})$$

$Y(e^{j\Omega})$ is 2π periodic.

■ C.9.3 SAMPLING THE DTFT IN FREQUENCY

$$w[n] = \sum_{m=-\infty}^{\infty} v[n + mN] \xleftrightarrow{\text{DTFS}; \Omega_0=2\pi/N} W[k] = \frac{1}{N} V(e^{jk\Omega_0})$$

$w[n]$ is N periodic.

■ C.9.4 SAMPLING THE FT IN FREQUENCY

$$g(t) = \sum_{m=-\infty}^{\infty} x(t + mT) \xleftrightarrow{\text{FS}; \omega_0=2\pi/T} G[k] = \frac{1}{T} X(jk\omega_0)$$

$g(t)$ is T periodic.