# CS 331: Artificial Intelligence Fundamentals of Probability III

Thanks to Andrew Moore for some course material

#### Full Joint Probability Distributions

The probabilities

Coin	Card	Candy	P(Coin, Card, Candy)			
tails	black	1	0.15			
tails	black	2	0.06			
tails	black	3	0.09			
tails	red	1	0.02			
tails	red	2	0.06			
tails	red	3	0.12			
heads	black	1	0.075			
heads	black	2	0.03			
heads	black	3	0.045			
heads	red	1	0.035			
heads	red	2	0.105			
heads	red	3	0.21			

This cell means P(Coin=heads, Card=red, Candy=3) = 0.21

### Marginalization

The general marginalization rule for any **sets** of variables *Y* and *Z*:

$$P(Y) = \sum_{\mathbf{z}} P(Y, \mathbf{z})$$

or

z is over all possible combinations of values of Z (remember Z is a set)

$$P(Y) = \sum_{\mathbf{z}} P(Y \mid \mathbf{z}) P(\mathbf{z})$$

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#### **Conditional Probabilities**

We can also compute conditional probabilities from the joint. Recall:

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

#### Inference

We will write the query as  $P(X \mid e)$ 

$$P(X | e) = \alpha P(X, e) = \alpha \sum P(X, e, y)$$

Summation is over all possible combinations of values of the unobserved variables *Y* 

X = Query variable (a single variable for now)

E = Set of evidence variables

e = the set of observed values for the evidence variables

Y = Unobserved variables

# Bayes' Rule

The product rule can be written in two ways:

$$P(A, B) = P(A \mid B)P(B)$$

$$P(A, B) = P(B \mid A)P(A)$$

You can combine the equations above to get:

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

### Bayes' Rule



More generally, the following is known as Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Note that these are distributions

Sometimes, you can treat P(B) as a normalization constant  $\alpha$ 

$$P(A \mid B) = \alpha P(B \mid A) P(A)$$

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#### More General Forms of Bayes Rule

If A takes 2 values:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|-A)P(-A)}$$

If A takes  $n_A$  values:

$$P(A = v_i | B) = \frac{P(B | A = v_i)P(A = v_i)}{\sum_{k=1}^{n_A} P(B | A = v_k)P(A = v_k)}$$

### When is Bayes Rule Useful?

Sometimes it's easier to get P(X|Y) than P(Y|X).

Information is typically available in the form  $P(effect \mid cause \mid effect)$ 

For example, P( symptom | disease ) is easy to measure empirically but obtaining P( disease | symptom ) is harder

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### Bayes Rule Example

Meningitis causes stiff necks with probability 0.5. The prior probability of having meningitis is 0.00002. The prior probability of having a stiff neck is 0.05. What is the probability of having meningitis given that you have a stiff neck?

Let m =patient has meningitis

Let s = patient has stiff neck

P(s | m) = 0.5

P(m) = 0.00002

P(s) = 0.05

$$P(m \mid s) = \frac{P(s \mid m)P(m)}{P(s)} = \frac{(0.5)(0.00002)}{0.05} = 0.0002$$

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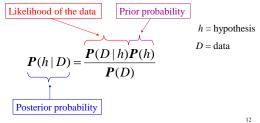
 $P(s \mid m) = 0.5$ P(m) = 0.00002 Note: Even though P(s|m) = 0.5, P(m|s) = 0.0002

P(s) = 0.05

 $P(m \mid s) = \frac{P(s \mid m)P(m)}{P(s)} = \frac{(0.5)(0.00002)}{0.05} = 0.0002$ 

# How is Bayes Rule Used

In machine learning, we use Bayes rule in the following way:



### Bayes Rule With More Than One Piece of Evidence

Suppose you now have 2 evidence variables Card=red and Candy=1 (note that Coin is uninstantiated below)

 $P(Coin \mid Card = red, Candy = 1)$ =  $\alpha P(Card = red, Candy = 1 \mid Coin) P(Coin)$ 

In order to calculate  $P(Card=red, Candy=1 \mid Coin)$ , you need a table of 6 probability values. With N Boolean evidence variables, you need  $2^N$  probability values.

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### Independence

We say that variables X and Y are independent if any of the following hold: (note that they are all equivalent)

$$P(X | Y) = P(X)$$
 or

$$P(Y \mid X) = P(Y)$$
 or

$$P(X,Y) = P(X)P(Y)$$

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### Why is independence useful?

P(Card, Candy) = P(Card)P(Candy)

This table has 2 values

This table has 3 values

- You now need to store 5 values to calculate *P*(*Coin*, *Card*, *Candy*)
- · Without independence, we needed 6

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### Conditional Independence

Suppose I tell you that to select a piece of *Candy*, I first flip a *Coin*. If heads, I select a *Card* from one (stacked) deck; if tails, I select from a different (stacked) deck. The color of the card determines the bag I select the *Candy* from, and each bag has a different mix of the types of *Candy*.

Are Coin and Candy independent?

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### Conditional Independence

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Are Coin and Candy independent? No.

But given Card, they are independent!

$$P(Coin = heads, Candy = 3 | Card) = P(Coin = heads | Card) \times P(Candy = 3 | Card)$$

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# Conditional Independence

General form:

$$P(A, B \mid C) = P(A \mid C)P(B \mid C)$$

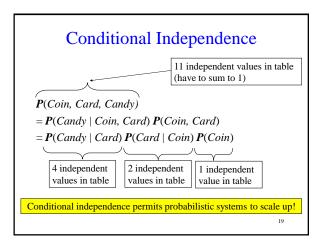
Or equivalently:

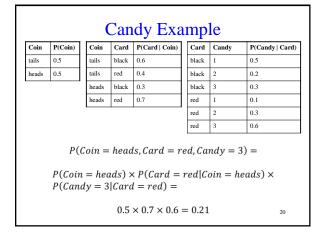
$$P(A \mid B, C) = P(A \mid C)$$
 and

$$P(B \mid A, C) = P(B \mid C)$$

How to think about conditional independence:

In  $P(A \mid B, C) = P(A \mid C)$ : if knowing C tells me everything about A, I don't gain anything by knowing B





	Practice											
Coin	P(Coin)	Coin	Card	P(Card   Coin)	Card	Candy	P(Candy   Card)					
tails	0.5	tails	black	0.6	black	1	0.5					
heads	0.5	tails	red	0.4	black	2	0.2					
		heads	black	0.3	black	3	0.3					
		heads	red	0.7	red	1	0.1					
red 2							0.3					
red							0.6					
	C	ompute	e P(Co	oin = tails	Card	= red)						
							21					

#### What You Should Know

- How to do inference in joint probability distributions
- How to use Bayes Rule
- Why independence and conditional independence is useful