

CS 331: Artificial Intelligence

Probability I

Thanks to Andrew Moore for some course material

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Dealing with Uncertainty

- We want to get to the point where we can reason with **uncertainty**
- This will require using probability e.g. probability that it will rain today is 0.99
- We will review the fundamentals of probability

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Outline

1. Random variables
2. Probability

Random Variables

- The basic element of probability is the **random variable**
- Think of the random variable as an event with some degree of uncertainty as to whether that event occurs
- Random variables have a **domain** of values it can take on

Random Variables

Example:

- *ProfLate* is a random variable for whether your prof will be late to class or not
- The domain of *ProfLate* is $\{true, false\}$
 - *ProfLate* = *true*: proposition that prof will be late to class
 - *ProfLate* = *false*: proposition that prof will not be late to class

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Random Variables

Example:

- *ProfLate* is a random variable for whether your prof will be late to class or not
- The domain of *ProfLate* is $\langle true, false \rangle$
 - *ProfLate* = *true*: proposition that prof will be late to class
 - *ProfLate* = *false*: proposition that prof will not be late to class

You can assign some degree of belief to this proposition e.g.
 $P(\text{ProfLate} = \text{true}) = 0.9$

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Random Variables

Example:

- *ProfLate* is a random variable for whether your prof will be late to class or not
- The domain of *ProfLate* is $\langle true, false \rangle$
 - *ProfLate* = *true*: proposition that prof will be late to class
 - *ProfLate* = *false*: proposition that prof will not be late to class

And to this one e.g.
 $P(\text{ProfLate} = \text{false}) = 0.1$

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Random Variables

- We will refer to **random variables** with **capitalized names** e.g. *X*, *Y*, *ProfLate*
- We will refer to **names of values** with **lower case names** e.g. *x*, *y*, *proflate*
- This means you may see a statement like $\text{ProfLate} = \text{proflate}$
 - This means the random variable *ProfLate* takes the value *proflate* (which can be *true* or *false*)
- Shorthand notation:
 $\text{ProfLate} = \text{true}$ is the same as *proflate* and
 $\text{ProfLate} = \text{false}$ is the same as $\neg \text{proflate}$

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Random Variables

3 types of random variables:

1. Boolean random variables
2. Discrete random variables
3. Continuous random variables

Boolean Random Variables

- Take the values *true* or *false*
- E.g. Let A be a Boolean random variable
 - $P(A = \text{false}) = 0.9$
 - $P(A = \text{true}) = 0.1$

Discrete Random Variables

Allowed to take on a finite number of values

e.g.

- $P(\text{DrinkSize}=\textit{small}) = 0.1$
- $P(\text{DrinkSize}=\textit{medium}) = 0.2$
- $P(\text{DrinkSize}=\textit{large}) = 0.7$

Discrete Random Variables

Values of the domain must be:

- Mutually Exclusive i.e. $P(A = v_i \text{ AND } A = v_j) = 0$ if $i \neq j$

This means, for instance, that you can't have a drink that is both *small* and *medium*

- Exhaustive i.e. $P(A = v_1 \text{ OR } A = v_2 \text{ OR } \dots \text{ OR } A = v_k) = 1$

This means that a drink can only be either *small*, *medium* or *large*. There isn't an *extra large*.

Discrete Random Variables

Values of the domain must be:

- Mutually Exclusive i.e. $P(A = v_i \text{ AND } A = v_j) = 0$ if $i \neq j$

This means, for i
drink that is both

The AND here means intersection
i.e. $(A = v_i) \cap (A = v_j)$

- Exhaustive i.e. $P(A = v_1 \text{ OR } A = v_2 \text{ OR } \dots \text{ OR } A = v_k) = 1$

This means that
medium or large

The OR here means union i.e. $(A = v_1) \cup (A = v_2) \cup \dots \cup (A = v_k)$

Discrete Random Variables

- Since we now have multi-valued discrete random variables we can't write $P(a)$ or $P(\neg a)$ anymore
- We have to write $P(A = v_i)$ where $v_i =$ a value in $\{v_1, v_2, \dots, v_k\}$

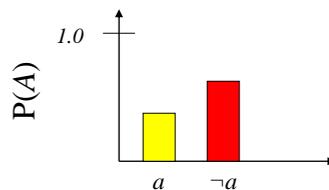
Continuous Random Variables

- Can take values from the real numbers
- E.g. They can take values from $[0, 1]$
- **Note: We will primarily be dealing with discrete random variables**
- (The next slide is just to provide a little bit of information about continuous random variables)

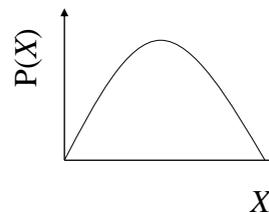
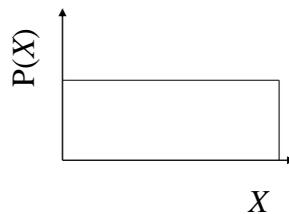
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Probability Density Functions

Discrete random variables have probability distributions:



Continuous random variables have probability density functions e.g:



Probabilities

- We will write $P(A=true)$ as “the fraction of possible worlds in which A is true”
- We can debate the philosophical implications of this for the next 4 hours
- But we won't

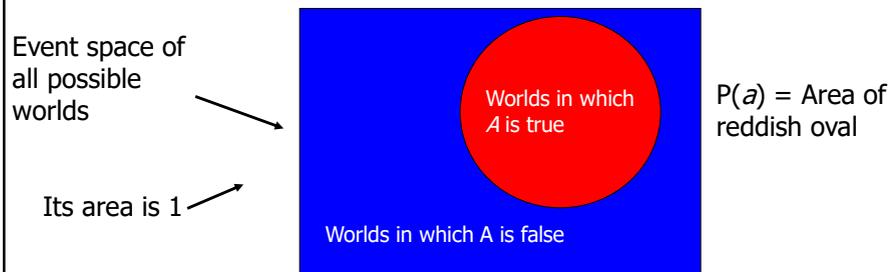
Probabilities

- We will sometimes talk about the probabilities of all possible values of a random variable
- Instead of writing
 - $P(A=false) = 0.25$
 - $P(A=true) = 0.75$
- We will write $\mathbf{P}(A) = (0.25, 0.75)$



Note the boldface!

Visualizing A



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The Axioms of Probability

- $0 \leq P(a) \leq 1$
- $P(\text{true}) = 1$
- $P(\text{false}) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a \text{ AND } b)$

↑
The logical OR is equivalent to set union \cup .

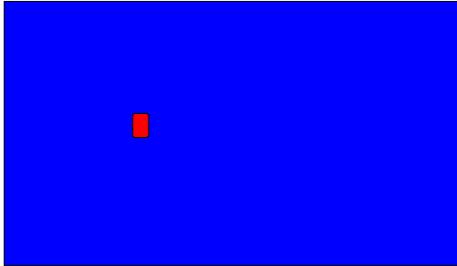
↑
The logical AND is equivalent to set intersection \cap . Sometimes, I'll write it as $P(a, b)$

These axioms are often called Kolmogorov's axioms in honor of the Russian mathematician Andrei Kolmogorov

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Interpreting the axioms

- $0 \leq P(a) \leq 1$
- $P(\text{true}) = 1$
- $P(\text{false}) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a, b)$



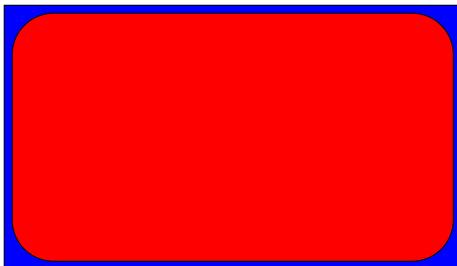
The area of $P(a)$ can't get any smaller than 0

And a zero area would mean that there is no world in which a is not false

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Interpreting the axioms

- $0 \leq P(a) \leq 1$
- $P(\text{true}) = 1$
- $P(\text{false}) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a, b)$



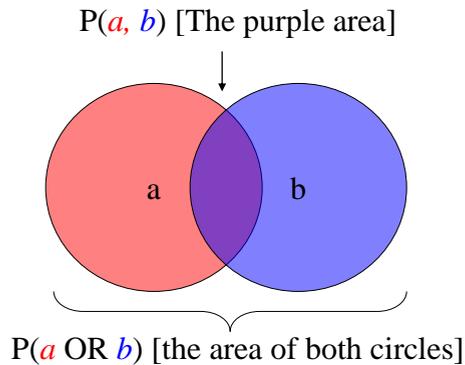
The area of $P(a)$ can't get any bigger than 1

And an area of 1 would mean all worlds will have a is true

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Interpreting the axioms

- $0 \leq P(a) \leq 1$
- $P(\text{true}) = 1$
- $P(\text{false}) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a, b)$



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Prior Probability

- We can consider $P(A)$ as the unconditional or **prior probability**
 - E.g. $P(\text{ProfLate} = \text{true}) = 1.0$
- It is the probability of event A in the absence of any other information
- If we get new information that affects A , we can reason with the **conditional probability** of A given the new information.

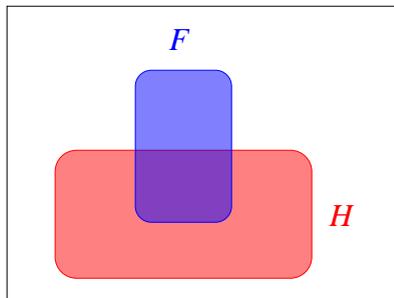
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Conditional Probability

- $P(A | B)$ = Fraction of worlds in which B is true that also have A true
- Read this as: “Probability of A conditioned on B ”
- Prior probability $P(A)$ is a special case of the conditional probability $P(A |)$ conditioned on no evidence

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Conditional Probability Example



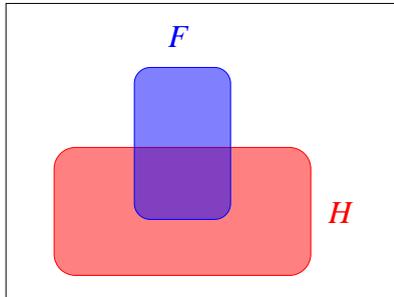
H = "Have a headache"
 F = "Coming down with Flu"

$P(H) = 1/10$
 $P(F) = 1/40$
 $P(H | F) = 1/2$

"Headaches are rare and flu is rarer, but if you're coming down with 'flu there's a 50-50 chance you'll have a headache."

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Conditional Probability



H = "Have a headache"
 F = "Coming down with Flu"

$$\begin{aligned}P(H) &= 1/10 \\P(F) &= 1/40 \\P(H | F) &= 1/2\end{aligned}$$

$P(H|F)$ = Fraction of flu-inflicted worlds in which you have a headache

$$\begin{aligned}&= \frac{\# \text{ worlds with flu and headache}}{\# \text{ worlds with flu}} \\&= \frac{\text{Area of "H and F" region}}{\text{Area of "F" region}} \\&= \frac{P(H,F)}{P(F)}\end{aligned}$$

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Definition of Conditional Probability

$$P(A | B) = \frac{P(A, B)}{P(B)}$$

Corollary: The Chain Rule (aka The Product Rule)

$$P(A, B) = P(A | B)P(B)$$

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Important Note

$$P(A | B) + P(\neg A | B) = 1$$

But:

$$P(A | B) + P(A | \neg B) \text{ does not always } = 1$$

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The Joint Probability Distribution

- $P(A, B)$ is called the joint probability distribution of A and B
- It captures the probabilities of all combinations of the values of a set of random variables

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The Joint Probability Distribution

- For example, if A and B are Boolean random variables, then $P(A,B)$ could be specified as:

$P(A=false, B=false)$	0.25
$P(A=false, B=true)$	0.25
$P(A=true, B=false)$	0.25
$P(A=true, B=true)$	0.25

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The Joint Probability Distribution

- Now suppose we have the random variables:
 - $Drink = \{coke, sprite\}$
 - $Size = \{small, medium, large\}$
- The joint probability distribution for $P(Drink, Size)$ could look like:

$P(Drink=coke, Size=small)$	0.1
$P(Drink=coke, Size=medium)$	0.1
$P(Drink=coke, Size=large)$	0.3
$P(Drink=sprite, Size=small)$	0.1
$P(Drink=sprite, Size=medium)$	0.2
$P(Drink=sprite, Size=large)$	0.2

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Full Joint Probability Distribution

- Suppose you have the complete set of random variables used to describe the world
- A joint probability distribution that covers this complete set is called the **full joint probability distribution**
- Is a complete specification of one's uncertainty about the world in question
- Very powerful: Can be used to answer any probabilistic query

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