CS 331: Artificial Intelligence
Propositional Logic 2

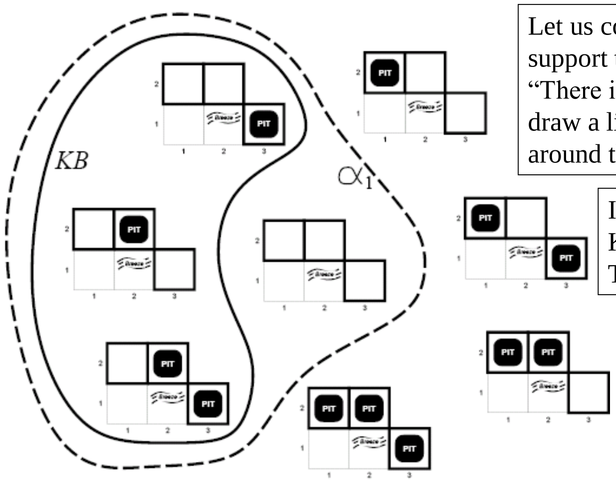
1

Review of Last Time

- |= means "logically follows"
- $|-_i$ means "can be derived from"
- If your inference algorithm derives only things that follow logically from the KB, the inference is sound
- If everything that follows logically from the KB can be derived using your inference algorithm, the inference is complete

2

Entailment Applied to the Wumpus World



Let us consider the models that support the conclusion α_1 = "There is no pit in [1,2]." We draw a line marked with α_1 around these models

In every model in which KB is true, α_1 is also true. Therefore KB |= α_1

Inference: Model Checking

- Suppose we want to know if KB |= $\neg P_{1,2}$?
- In the 3 models in which KB is true, $\neg P_{1,2}$ is also true

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
false	false	false	false	false	false	false	true	true	true	true	false	false
false	false	false	false	false	false	true	true	true	false	true	false	false
:	:	:	:	:	:	:	:	:	:	:	:	:
false	true	false	false	false	false	false	true	true	false	true	true	false
false	true	false	false	false	false	true	true	true	true	true	true	true
false	true	false	false	false	true	false	true	true	true	true	true	true
false	true	false	false	false	true	true	true	true	true	true	true	true
false	true	false	false	true	false	false	true	false	false	true	true	false
:	:	:	:	:	:	:	:	:	:	:	:	:
true	true	true	true	true	true	true	false	true	true	false	true	false

Complexity

- If the KB and α contain n symbols in total, what is the time complexity of the truth table enumeration algorithm?

- Space complexity is O(n) because the actual algorithm uses DFS

5

The really depressing news

- Every known inference algorithm for propositional logic has a **worst-case** complexity that is exponential in the size of the input

- But some algorithms are more efficient **in practice**

6

Logical equivalence

- Intuitively: two sentences α and β are logically equivalent (i.e. $\alpha \equiv \beta$) if they are true in the same set of models
- Formally: $\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$
- Can prove this with truth tables

Standard Logic Equivalences

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) && \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) && \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) && \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) && \text{associativity of } \vee \\
\neg(\neg \alpha) &\equiv \alpha && \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg \beta \Rightarrow \neg \alpha) && \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg \alpha \vee \beta) && \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) && \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg \alpha \vee \neg \beta) && \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg \alpha \wedge \neg \beta) && \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) && \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) && \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

In the above, α, β, and γ are arbitrary sentences of propositional logic

Validity

- A sentence is valid if it is true in all models
- E.g. $P \vee \neg P$ is valid
- Valid sentences = Tautologies
- Tautologies are vacuous

Deduction theorem

For any sentences α and β, $\alpha \models \beta$ iff the sentence $(\alpha \Rightarrow \beta)$ is valid

9

Satisfiability

- A sentence is satisfiable if it is true in some model.
- A sentence is unsatisfiable if it is true in no models
- Determining the satisfiability of sentences in propositional logic was the first problem proved to be NP-complete
- Satisfiability is connected to validity:
  α is valid iff $\neg\alpha$ is unsatisfiable
- Satisfiability is connected to entailment:
  $\alpha \models \beta$ iff the sentence $(\alpha \wedge \neg\beta)$ is unsatisfiable (proof by contradiction)

10

CW: Exercise

- Is the following sentence valid?

$$(A \Rightarrow B) \lor (\neg A \Rightarrow \neg B)$$

Proof methods

How do we prove that α can be entailed from the KB?

1. Model checking e.g. check that α is true in all models in which KB is true
2. Inference rules

Inference Rules

1. Modus Ponens

$$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$

2. And-Elimination

$$\frac{\alpha \wedge \beta}{\alpha}$$

These are both sound inference rules. You don't need to enumerate models now

13

Other Inference Rules

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) \quad \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) \quad \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) \quad \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

All of the logical equivalences can be turned into inference rules e.g.

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}$$

Example

Given the following KB, can we prove ¬R?

KB:

P ⇒ ¬(Q ∨ R)

P

Proof:

¬(Q ∨ R) by Modus Ponens

¬Q ∧ ¬R by De Morgan's Law

¬R by And-Elimination

Proofs

- A sequence of applications of inference rules is called a proof
- Instead of enumerating models, we can search for proofs
- Proofs ignore irrelevant propositions
- 2 methods:
  - Go forward from initial KB, applying inference rules to get to the goal sentence
  - Go backward from goal sentence to get to the KB

In-class Exercise

If it is October, there will not be a football game at OSU	
If it is October and it is Saturday, I will be in Corvallis	
If it doesn't rain or if there is a football game, I will ride my bike to OSU	
Today is Saturday and it is October	
If I am in Corvallis, it will not rain	

Can you prove that I will ride my bike to OSU?

Monotonicity

- Proofs only work because of monotonicity
- Monotonicity: the set of entailed sentences can only increase as information is added to the knowledge base
- For any sentences α and β,

  if KB $\models \alpha$ then KB $\wedge\ \beta \models \alpha$

Resolution

- An inference rule that is sound and complete
- Forms the basis for a family of complete inference procedures
- Here, complete means refutation completeness: resolution can refute or confirm the truth of any sentence with respect to the KB

19

Resolution

- Here's how resolution works ($\neg l_2$ and l_2 are called complementary literals):

$$\frac{l_1 \vee l_2, \quad \neg l_2 \vee l_3}{l_1 \vee l_3}$$

- Note that you need to remove multiple copies of literals (called factoring) i.e.

$$\frac{l_1 \vee l_2, \quad \neg l_2 \vee l_1}{l_1}$$

- If l_i and m_j are complementary literals, the full resolution rule looks like:

$$\frac{l_1 \vee \cdots \vee l_k, \quad m_1 \vee \cdots \vee m_n}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

Conjunctive Normal Form

- Resolution only applies to sentences of the form $l_1 \vee l_2 \vee \ldots \vee l_k$
- This is called a disjunction of literals
- It turns out that every sentence of propositional logic is logically equivalent to a conjunction of disjunction of literals
- Called Conjunctive Normal Form or CNF

  e.g. $(l_1 \vee l_2 \vee l_3 \vee l_4) \wedge (l_5 \vee l_6 \vee l_7 \vee l_8) \wedge \ldots$
- k-CNF sentences have exactly k literals per clause e.g. A 3-CNF sentence would be $(l_1 \vee l_2 \vee l_3) \wedge (l_4 \vee l_5 \vee l_6) \wedge (l_7 \vee l_8 \vee l_9)$

Recipe for Converting to CNF

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$
3. Move $\neg$ inwards using:

   $\neg(\neg\alpha) \equiv \alpha$ (double-negation elimination)

   $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$ (De Morgan's Law)

   $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$ (De Morgan's Law)
4. Apply distributive law $(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

In-class Exercise

KB

Person ⇒ Mortal
Socrates ⇒ Person

Can we show that :

KB |= (Socrates ⇒ Mortal)?

23

Exercise

- Convert the following sentence to CNF.

$$(B \vee C) \Rightarrow D$$

24

A resolution algorithm

To prove KB $\models \alpha$, we show that (KB $\wedge \neg\alpha$) is unsatisfiable
(Remember that $\alpha \models \beta$ iff the sentence ($\alpha \wedge \neg\beta$) is unsatisfiable)

The algorithm:

1. Convert (KB $\wedge \neg\alpha$) to CNF
2. Apply resolution rule to resulting clauses. Each pair with complementary literals is resolved to produce a new clause which is added to the KB
3. Keep going until
   - There are no new clauses that can be added (meaning KB $\not\models \alpha$)
   - Two clauses resolve to yield the empty clause (meaning KB $\models \alpha$)

The empty clause is equivalent to false because a disjunction is true only if one of its disjuncts is true

In-class Exercise

KB

Person $\Rightarrow$ Mortal
Socrates $\Rightarrow$ Person

Can we show that :

KB $\models$ (Socrates $\Rightarrow$ Mortal)?

CW: Exercise

- Suppose the KB contains the following sentences in CNF.
  1. $\neg C \lor E$
  2. $\neg P \lor E$
  3. $\neg E \lor \neg R$
  4. $\neg A \lor \neg P \lor E$
- Does $KB \models \neg R$?

27

Resolution Pseudocode

```
function PL-RESOLUTION(KB, α) returns true or false
   clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
   new ← { }
   loop do
      for each C_i, C_j in clauses do
         resolvents ← PL-RESOLVE(C_i, C_j)
         if resolvents contains the empty clause then return true
         new ← new ∪ resolvents
      if new ⊆ clauses then return false
      clauses ← clauses ∪ new
```

28

Things you should know

- Understand the syntax and semantics of propositional logic
- Know how to do a proof in propositional logic using inference rules
- Know how to convert arbitrary sentences to CNF
- Know how resolution works

29