Homework 2 solution

Problem 1

a)

$$\begin{aligned} P(x_1) &= 1 - P(x_0) = 1 - 0.5 = 0.5. \\ P(y_0|x_0) &= q_0 = 1 - p_0 = 1 - 0.2 = 0.8. \\ P(y_1|x_1) &= q_1 = 1 - p_1 = 1 - 0.3 = 0.7. \end{aligned} \quad (3)$$

Using total probability, we get

$$\begin{aligned} P(y_0) =& P(y_0|x_0)P(x_0) + P(y_0|x_1)P(x_1) \\ =& q_0 P(x_0) + p_1 P(x_1) \\ =& (0.8 \times 0.5) + (0.3 \times 0.5) \\ =& 0.55. \end{aligned}$$

$$\begin{aligned} P(y_1) =& P(y_1|x_0)P(x_0) + P(y_1|x_1)P(x_1) \\ =& p_0 P(x_0) + q_1 P(x_1) \\ =& (0.2 \times 0.5) + (0.7 \times 0.5) \\ =& 0.45. \end{aligned}$$

b) Using Bayes rule, we get

$$P(x_0|y_0) = \frac{P(x_0)P(y_0|x_0)}{P(y_0)} = \frac{0.5 \times 0.8}{0.55} = 0.727.$$

c) Similarly, applying Bayes rule we get

$$P(x_1|y_1) = \frac{P(x_1)P(y_1|x_1)}{P(y_1)} = \frac{0.5 \times 0.7}{0.45} = 0.78.$$

d) The total error is given by

$$P_e = P(y1|x0)P(x0) + P(y0|x1)P(x1) = (0.2 \times 0.5) + (0.3 \times 0.5) = 0.25.$$

Problem 2

Given,

$$\begin{aligned} &P(\text{Output from Machine I}) = 0.8, \\ &P(\text{Output from Machine II}) = 0.1, \\ &P(\text{Output from Machine III}) = 0.1. \end{aligned}$$

$$\begin{aligned} &P(\text{Output is defective } |\text{Output from Machine I}) = 0.05 \\ &P(\text{Output is defective } |\text{Output from Machine II}) = 0.1 \\ &P(\text{Output is defective } |\text{Output from Machine III}) = 0.15. \end{aligned}$$

$$\begin{aligned} P(\text{Output is defective}) &= P(\text{Output is defective } |\text{Output from Machine I})P(\text{Output from Machine I}) \\ &+ P(\text{Output is defective } |\text{Output from Machine II})P(\text{Output from Machine II}) \\ &+ P(\text{Output is defective } |\text{Output from Machine III})P(\text{Output from Machine III}) \\ &= (0.8 \times 0.05) + (0.1 \times 0.1) + (0.1 \times 0.15) = 0.04 + 0.01 + 0.015 = 0.065 \end{aligned}$$

6.5% of the total products are defective.

Problem 3

As A, B are independent, we have

$$P(A \cap B) = P(A)P(B). \tag{4}$$

$$\begin{aligned}
P(\bar{A} \cap \bar{B}) =& P(\overline{A \cup B}) \\
=& 1 - P(A \cup B) \\
=& 1 - P(A) - P(B) + P(A \cap B) \\
=& 1 - P(A) - P(B) - P(A)P(B), \quad \text{because } A, B \text{ are independent} \\
=& [1 - P(A)] - P(B)[1 - P(A)] \\
=& [1 - P(A)][1 - P(B)] \\
=& P(\bar{A})P(\bar{B}).
\end{aligned}$$

Problem 4

Let W be the event that a white ball is drawn, and let H be the event that the coin comes up heads. Moreover, let T to be the event that the coin comes up tails. Then, the desired probability is given by

$$\begin{aligned}
P(H|W) &= \frac{P(H \cap W)}{P(W)} \\
&= \frac{P(W|H)P(H)}{P(W)} \\
&= \frac{P(W|H)P(H)}{P(W|H)P(H) + P(W|T)P(T)} \\
&= \frac{\frac{2}{9}\frac{1}{2}}{\frac{2}{9}\frac{1}{2} + \frac{5}{11}\frac{1}{2}} = \frac{22}{67}.
\end{aligned}$$