ECE 353 Probability and Random Signals - Homework 5

Spring 2019
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Due: May. 7, 2019

Q1. When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is the PMF of N, the number of times the system sends the same message?

(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

Solution 1

(a) The paging message is sent again and again until a success occurs. Hence the number of paging messages is $N = n$ if there are $n-1$ paging failures followed by a paging success. That is, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p, & n = 1, 2, ... \\ 0, & otherwise \end{cases} \tag{1}$$

(b) The probability that no more three paging attempts are required is

$$P[N \leq 3] = 1 - P[N > 3] = 1 - \sum_{n=4}^{\infty} P_N(n) = 1 - (1-p)^3 \tag{2}$$

This answer can be obtained without calculation since $N > 3$ if the first three paging attempts fail and that event occurs with probability $(1-p)^3$. Hence, we must choose p to satisfy $1 - (1-p)^3 \geq 0.95$ or $(1-p)^3 \leq 0.005$. This implies

$$p \geq 1 - (0.05)^{1/3} \approx 0.6316 \tag{3}$$

Q2. The number of buses is B. The number of buses that arrive at a bus stop in T minutes is a Poisson random variable with the parameter λT and $\lambda = 1/5$.

(a) What is the PMF of B, the number of buses that arrive in T minutes?
(b) What is the probability that in a two-minute interval, three buses will arrive?
(c) What is the probability of no buses arriving in a 10-minute interval?

Solution 2 Since an average of $T/5$ buses arrive in an interval of T minutes, buses arrive at the bus stop at a rate of $1/5$ buses per minute.

(a) From the definition of the Poisson PMF, the PMF of B, the number of buses in T minutes, is

$$P_B(b) = \begin{cases} (T/5)^b e^{-T/5}/b!, & b = 0, 1, \cdots \\ 0, & otherwise \end{cases} \tag{4}$$

(b) Choosing $T = 2$ minutes, the probability that three buses arrive in a two minute interval is

$$P_B(3) = (2/5)^3 e^{-2/5}/3! \approx 0.0072 \tag{5}$$

(c) By choosing $T = 10$ minutes, the probability of zero buses arriving in a ten minute interval is

$$P_B(0) = e^{-10/5}/0! = e^{-2} \approx 0.135 \tag{6}$$

Q3. (a) Consider you go fishing. Each time you cast your line, your hook will be swallowed by a fish with probability h, independent of any other casts. What is the PMF of K, the number of fish hooked after m casts?

(b) Consider the same setup as in (a). Suppose that you cast for 6 times and each time your hook will be swallowed by a fish with probability 0.3 . What is the probability that the number of fish you caught is greater than or equal to 4?

Solution 3

(a) Whether a hook catches a fish is an independent trial with success probability h. The the number of fish hooked K, has the binomial PMF

$$P_K(k) = \begin{cases} \binom{m}{k}(h)^k(1-h)^{m-k}, & k = 0, 1, \ldots, m \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

(b) The probability that you catch more than 4 fish (including 4) is:

$$\begin{aligned} P[\text{catch more than 4 fish}] &= P_K(4) + P_K(5) + P_K(6) \\ &= \binom{6}{4}(0.3)^4(1-0.3)^2 + \binom{6}{5}(0.3)^5(1-0.3)^1 \\ &\quad + \binom{6}{6}(0.3)^6(1-0.3)^0 \\ &= 0.0705 \end{aligned}$$

Q4. The peak temperature T on any day in Antarctica in July is a Gaussian random variable with a variance of 225. With probability 1/2, the temperature T exceeds 10 degrees. What is $P[T > 32]$, the probability the temperature is above freezing? (Hint: here is an example of using the table. If you wish to know $\Phi(1.12)$, you should use the value on the row corresponding to 1.1 and the column corresponding to +0.02, i.e., $\Phi(1.12)$=0.86864.) (15%)

z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189

Table 1: Part of standard normal table

Solution

We are given that T is a Gaussian random variable with variance 225. We also know that with probability 1/2, T exceeds 10 degrees. First we would like to find the mean temperature, and we do so by looking at the second fact.

$$P[T > 10] = 1 - P[T \leq 10] = 1 - \Phi(\frac{10 - \mu_T}{15}) = 1/2 \tag{8}$$

By looking at the table we find that if $\Phi(\Gamma) = 1/2$, then $\Gamma = 0$. Therefore,

$$\Phi(\frac{10 - \mu_T}{15}) = 1/2$$

which implies that $(10 - \mu_T)/15 = 0$ or $\mu_T = 10$. Now we have a Gaussian T with mean 10 and standard deviation 15. So we are prepared to answer the following problems.

$$\begin{aligned} P[T > 32] &= 1 - P[T \leq 32] = 1 - \Phi(\frac{32 - 10}{15}) \\ &= 1 - \Phi(1.45) = 1 - 0.926 = 0.074 \end{aligned}$$

Q5. Let $X \sim \text{Exp}(\lambda)$ and $Y = \lfloor X \rfloor$.

(a) Determine whether Y is more likely to be even or odd?

(b) For what value of λ, the probability that Y is even is two times the probability that Y is odd.

Solution 5

(a) For $X \sim \text{Exp}(\lambda)$, the pdf of X is given

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases} \tag{9}$$

for $Y = \lfloor X \rfloor$, the probability that Y is even is given by

$$\sum_{k=0}^{\infty} P\left(2k \leq X \leq 2k+1\right) = \sum_{k=0}^{\infty} P\left(2k \leq X \leq 2k+1\right) \tag{10}$$

$$= \sum_{k=0}^{\infty} \int_{2k}^{2k+1} \lambda e^{-\lambda x} dx = \frac{1}{1+e^{-\lambda}} \tag{11}$$

since $\lambda > 0$, the $\frac{1}{1+e^{-\lambda}} > \frac{1}{2}$, so the Y is more likely to be even.

(b) The probability of even Y is even is two times of the probability of even Y is odd.

$$\frac{1}{1+e^{-\lambda}} = 2(1 - \frac{1}{1+e^{-\lambda}}) \tag{12}$$

we obtain $\lambda = \ln 2$