ECE 353 : Probability and Random Signals
Homework 6
Spring 2019

Due May 16, 2019

1. Suppose X is a uniform random variable on the interval (0,1) and $Y = 5X + 2$.
   (a) Find the CDF of Y.
   (b) Find the PDF of Y and sketch it.
2. Let X be a geometric random variable with parameter p and n be a nonnegative integer. For what value of n is $P(X = n)$ maximum? What is the probability that X is odd?
3. Let X be uniformly distributed on $(-\pi, \pi)$ and $Y = \cos(X)$. Find the PDF of Y.
4. Let X be a continuous random variable with cdf $F_X(x)$. Let $Y = F_X(x)$. Show that Y is a uniform random variable over (0,1).

Homework 6 solution

Problem 1

a) As X is a uniform random variable, it's PDF is as follows.

$$f_X(x) = 1, \quad 0 < x < 1.$$

The CDF of X will be $F_X(x) = \begin{cases} 0, & x \leq 0 \\ \int_0^x f_X(x)dx = \int_0^x dx = x, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$.

The CDF of Y is as follows.

$$\begin{aligned} F_Y(y) &= p(Y \leq y), \\ &= p(5X + 2 \leq y), \\ &= p\left(X \leq \frac{y-2}{5}\right), \\ &= F_X\left(\frac{y-2}{5}\right), \\ &= \begin{cases} 0, & \frac{y-2}{5} < 0 \Rightarrow y \leq 2, \\ \frac{y-2}{5}, & 0 < \frac{y-2}{5} < 1 \Rightarrow 2 < y < 7, \\ 1, & \frac{y-2}{5} \geq 1 \Rightarrow y \geq 7. \end{cases} \end{aligned}$$

b) We know that the PDF of Y is $f_Y(y) = \frac{dF_Y(y)}{dy}$. Therefore, from (1) we get

$$f_Y(y) = \begin{cases} 0, & y \leq 2, \\ \frac{1}{5}, & 2 < y < 7, \\ 1, & \frac{y-2}{5} \geq 1 \Rightarrow y \geq 7. \end{cases}$$

Problem 2

a) As X has a geometric distribution. We have

$$p(X = n) = (1 - p)^{n-1} p \tag{1}$$

Since $0 \leq p \leq 1$, we have $0 \leq 1 - p \leq 1$. Then, we have

$$(1 - p)^{n-1} \leq 1, \quad \forall n \geq 1.$$

Equality holds only when $n = 1$, and the maximum value of (1) will be as follows.

$$p(x = n) = p.$$

b) If n is odd, we have $n = 2k+1$.

$$
\begin{aligned}
P(X \text{ is odd}) = \sum_{\text{n is odd}} p(X=n) &= \sum_{k=0}^{\infty} (1-p)^{2k+1-1} p \\
&= p \sum_{k=0}^{\infty} \{(1-p)^2\}^k \\
&= p\left(\frac{1}{1-(1-p)^2}\right) \\
&= p\left(\frac{1}{2p-p^2}\right) \\
&= \frac{1}{2-p}.
\end{aligned}
$$

Problem 3

As X has uniform distribution between $(-\pi, \pi)$,

$$
f_X(x) = \begin{cases} \frac{1}{2\pi}, & -\pi < x < \pi, \\ 0, & o.w. \end{cases}
$$

We know $f_Y(y) = \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|$

$$
\begin{aligned}
f_Y(y) =& \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|, \quad |y| < 1 \\
=& \sum_i f_X(\cos^{-1}(y)) \left| \frac{d\cos^{-1}(y)}{dy} \right| + \sum_i f_X(\pi - \cos^{-1}(y)) \left| \frac{d(\pi - \cos^{-1}(y))}{dy} \right|, \quad |y| < 1 \\
=& \frac{1}{2\pi} \frac{1}{\sqrt{1-y^2}} + \frac{1}{2\pi} \frac{1}{\sqrt{1-y^2}} \\
=& \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{1-y^2}}, & |y| < 1 \\ \\ 0. & o.w. \end{cases}
\end{aligned}
$$

Problem 4

We know from the properties of a cdf that $y = F_X(x)$ is a monotonically nondecreasing function. Since $0 \leq F_X(x) \leq 1$ for all real x, y in the interval (0,1).

$$
f_Y(y) = f_X(x) \frac{1}{\frac{dF_X(x)}{dx}} = \frac{f_X(x)}{f_X(x)} = 1, \quad 0 < y < 1.
$$

Hence, Y is a uniform random variable over (0,1).