ECE 353 Probability and Random Signals - Homework 7

Spring 2019
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Due: May 21, 2019

Q1. X is a continuous uniform (-5,5) random variable, i.e. $X \sim \mathrm{U}[-5,5]$

(a) Write the PDF $f_X(X)$?

(b) Compute $E[X]$ and $E[X^2]$.

(c) Compute $E[e^X]$.

Solution

(a) The PDF of a continuous uniform $(5,5)$ random variable is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) The expected value $E[X]$ is

$$\int_{-5}^{5} \frac{x}{10} dx = \left.\frac{x^2}{20}\right|_{-5}^{5} = 0$$

The expected value $E[X^2]$ is

$$\int_{-5}^{5} \frac{x^2}{10} dx = \left.\frac{x^3}{30}\right|_{-5}^{5} = \frac{25}{3}$$

(c) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10} dx = \left.\frac{e^x}{10}\right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84$$

Q2. Suppose that X, the inter arrival time between two packets from two different sources at a router, satisfies

$$P(x > t) = \alpha e^{-t} + \beta e^{-2t}, t \geq 0 \tag{1}$$

Where $\alpha + \beta = 1$ and $\beta \geq 0$. Calculate the mean of X.

Solution

First we can calculate

$$F_X(t) = 1 - P(x > t) = 1 - (\alpha e^{-t} + \beta e^{-2t}), t \geq 0 \tag{2}$$

we have

$$f_x(t) = \frac{\mathrm{d}F_X(t)}{\mathrm{d}t} = \alpha e^{-t} + 2\beta e^{-2t} \tag{3}$$

we can obtain

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt = \alpha \int_0^{\infty} t e^{-t} dt + 2\beta \int_0^{\infty} t e^{-2t} dt \tag{4}$$

$$= \alpha \int_0^{\infty} t e^{-t} dt + 2\beta \int_0^{\infty} \frac{s}{4} e^{-s} ds = \alpha + \frac{\beta}{2} \tag{5}$$

(Note: $\int_0^{\infty} t^n e^{-t} dt = n!$)

Q3. The random variable X has CDF

$$F_X(x) = \begin{cases} 0, & x < -3 \\ 0.4, & -3 \leq x < 5 \\ 0.8, & 5 \leq x < 7 \\ 1, & x \geq 7. \end{cases}$$

Let $B = \{X > 0\}$, find $P_{X|B}(x)$, $E[X|B]$ and $Var[X|B]$?

Solution

The PMF of X is:

$$P_X(x) = \begin{cases} 0.4, & x = -3 \\ 0.4, & x = 5 \\ 0.2, & x = 7 \\ 0, & \text{otherwise} \end{cases}$$

The event $B = \{X > 0\}$ has probability $P[B] = P_X(5) + P_X(7) = 0.6$, so the conditional PMF of X given B is:

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]}, & x \in B \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 2/3, & x = 5 \\ 1/3, & x = 7 \\ 0, & \text{otherwise} \end{cases}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_x x P_{X|B}(x) = 5 \times (2/3) + 7 \times (1/3) = 17/3 \tag{6}$$

$$E[X^2|B] = \sum_x x^2 P_{X|B}(x) = 5^2 \times (2/3) + 7^2 \times (1/3) = 33 \tag{7}$$

The conditional variance of X is

$$Var[X|B] = E[X^2|B] - (E[X|B])^2 = 33 - (17/3)^2 = 8/9 \tag{8}$$

Q4 $X \sim \mathrm{U}[0, 1]$ and $Y = -\log(X)$.

(a) Find the $f_Y(y)$.

(b) Compute $E[Y]$.

(c) Compute $E[Y^2]$.

(Hint: $\int_0^\infty t^n e^{-t} dt = n!$)

Solution

(a) From $Y = -\log(X)$, we can get $g^{-1}(y) = e^{-Y}$.

$$f_Y(y) = \begin{cases} f_X(e^{-y}) \left| \frac{de^{-y}}{dy} \right|, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

we obtain

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$E[Y] = E[-\log X] = -\int_0^1 \log x dx \tag{9}$$
$$= -(x\log x - x)|_0^1 = 1 \tag{10}$$

we can also have

$$E[Y] = \int_0^\infty y e^{-y} dy = 1 \tag{11}$$

(c)

$$E[Y^2] = \int_0^\infty y^2 e^{-y} dy = 2! = 2 \tag{12}$$