ECE 353 Probability and Random Signals - Homework 1

Spring 2019
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Q1. An experiment consists of tossing two six sided dice. Assume all outcomes have equal probability

(a) Find the sample space S.

(b) Find the probability of event A that the sum of the dots on the dice equals 6.

(c) Find the probability of event B that the sum of the dots on the dice is greater than 10.

(d) Find the probability of event C that the sum of the dots on the dice is greater than 8 but less than 12.

Solution 1

(a) The sample space is

$$\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix} \tag{1}$$

$$\tag{2}$$

(b)

$$A = \{(1,5) \quad (2,4) \quad (3,3) \quad (4,2) \quad (5,1)\} \tag{3}$$

$$\tag{4}$$

$P(A) = \frac{5}{36}$

(c)

$$B = \{(5,6) \quad (6,5) \quad (6,6)\} \tag{5}$$

$$\tag{6}$$

$P(A) = \frac{3}{36} = \frac{1}{12}$

(d)

$$C = \{(3,6) \quad (4,5) \quad (4,6) \quad (5,4) \quad (5,5) \quad (5,6) \quad (6,3) \quad (6,4) \quad (6,5)\} \tag{7}$$

$P(D) = \frac{9}{36} = \frac{1}{4}$

Q2. In an experiment, A, B, C and D are events with probabilities $P[A] = 1/4$, $P[B] = 1/8$, $P[C] = 5/8$, and $P[D] = 3/8$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find $P[A \cap B]$, $P[A \cup B]$, $P[A \cap \overline{B}]$, and $P[A \cup \overline{B}]$.

(b) Are A and B independent?

(c) Find $P[C \cap D]$, $P[C \cap \overline{D}]$, and $P[\overline{C} \cap \overline{D}]$.

(d) Are $\overline{C}$ and $\overline{D}$ independent?

Solution 2

(a) Since A and B are disjoint, $P[A \cap B] \ = 0$.
$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 3/8$.
It is obvious that $A \subset \overline{B}$ so that $A \cap \overline{B} = A$. This implies
$P[A \cap \overline{B}] = P[A] = 1/4$.
It also follows that $P[A \cup \overline{B}] = P[\overline{B}] = 1 - 1/8 = 7/8$.

(b) Events A and B are not independent since $P[A \cap B] \neq P[A]P[B]$.

(c) Since C and D are independent,
$P[C \cap D] = P[C]P[D] = 15/64$.
The next few items are a little trickier. We have $P[C \cap \overline{D}] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64$.
It follows that
$P[C \cup \overline{D}] = P[C] + P[D] - P[C \cup \overline{D}] = 5/8 + (1 - 3/8) - 25/64 = 55/64$.
Using De Morgan's law, we have
$P[\overline{C} \cap \overline{D}] = P[\overline{(C \cup D)}] = 1 - P[C \cup D] = 15/64$.

(d) Since $P[\overline{C} \cap \overline{D}] = P[\overline{C}]P[\overline{D}]$, $\overline{C}$ and $\overline{D}$ are independent.

Q3. Answer the following questions:

(a) Prove that $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ for any A and B (not necessarily disjoint).

(b) Prove that $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

Solution 3

(a) It can be easily checked that the sets A and $B \cap \overline{A}$ are a partition of $A \cup B$. Then, $(A \cup B) = A \cup (B \cap \overline{A})$ implies that $P[A \cup B] = P[A] + P[B \cap \overline{A}]$. Similarly, set B can be partitioned into sets $A \cap B$ and $B \cap \overline{A}$: $B = (A \cap B) \cup (B \cap \overline{A})$ meaning that $P[B] = P[A \cap B] + P[B \cap \overline{A}]$. Therefore,

$P[A \cup B] = P[A] + P[B \cap A^c] = P[A] + P[B] - P[A \cap B]$.

(b) In this question, we will be repeatedly using this axiom: if $X \cap Y = \varnothing$, then $P[X \cup Y] = P[X] + P[Y]$.

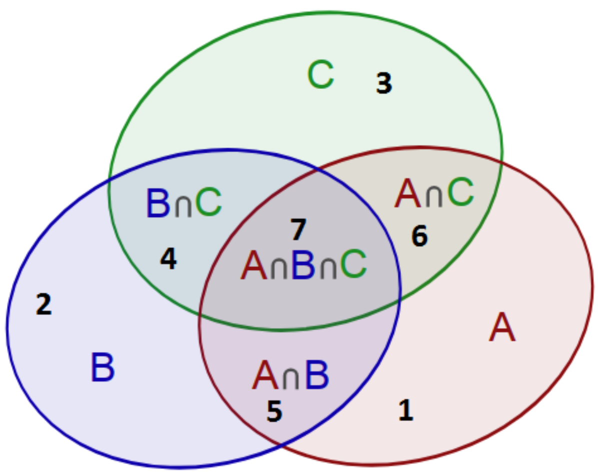

Figure 1: The union of three sets, cited from Wikipedia.

Let us consider the 7 disjoint subsets in Figure 1. The probability for each set is as follows:

1) $P[A] - P[A \cap C] - (P[A \cap B] - P[A \cap B \cap C])$
2) $P[B] - P[B \cap C] - (P[A \cap B] - P[A \cap B \cap C])$
3) $P[C] - P[B \cap C] - (P[A \cap C] - P[A \cap B \cap C])$
4) $P[B \cap C] - P[A \cap B \cap C]$
5) $P[A \cap B] - P[A \cap B \cap C]$
6) $P[A \cap C] - P[A \cap B \cap C]$

7) $P[A \cap B \cap C]$

By adding them together, the proof is completed.

Q4.

A number is selected uniformly at random from the set of integers $\{-100, -99, \ldots - 1, 0, 1, \ldots, 99, 100\}$ What is the probability that it is divisible by 11, but neither by 3 nor by 5?

Solution 4 The question can be simplified to What is the probability that it is divisible by 11, but neither by 3 nor by 5? Define three events as:

$$
\begin{aligned}
A &= \text{divisible by 3} && (8)\\
B &= \text{divisible by 5} && (9)\\
C &= \text{divisible by 11} && (10)\\
D &= \text{divisible by 11, but not by 3 and 5} && (11)\\
& && (12)
\end{aligned}
$$

$$P(C) = \frac{2 \times \lfloor 100/1 \rfloor + 1}{201} = \frac{19}{201} \quad (13)$$

$$P(C \cap A) = \frac{2 \times \lfloor 100/33 \rfloor + 1}{201} = \frac{7}{201} \quad (14)$$

$$P(C \cap B) = \frac{2 \times \lfloor 100/55 \rfloor + 1}{201} = \frac{3}{201} \quad (15)$$

$$P(C \cap A \cap B) = \frac{2 \times \lfloor 100/165 \rfloor + 1}{201} = \frac{1}{201} \quad (16)$$

$$P(D) = P(C) - P(C \cap A) - P(C \cap B) + P(C \cap A \cap B) = \frac{10}{201} \quad (17)$$