ECE 353 : Probability and Random Signals
Homework 4
Spring 2019

Due April 30, 2019

1. Random variable Y has a probability mass function (pmf) as

$$p_Y(y) = \begin{cases} \frac{c}{y}, & y = 1, 2 \\ \frac{c}{y^2}, & y = -1, -2 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value of the constant c.

   (b) Calculate

      i. $P(Y = -2)$

      ii. $P(Y < 1)$

2. Assume the resistance of R is a random variable, uniformly distributed on the interval $[850\Omega, 1150\Omega]$.

   (a) Find the PDF.

   (b) Calculate $P(900\Omega \leq 950\Omega)$?

3. In a restaurant, the time (in minutes) that a customer has to wait before s/he gets a table is specified by the following CDF:

$$F_X(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x \leq 1, \\ \frac{1}{2}, & 1 \leq x \leq 8, \\ \frac{x}{4} - \frac{3}{2}, & 8 \leq x \leq 10, \\ 1, & x \geq 10. \end{cases}$$

   (a) Compute and sketch the PDF $f_X(x)$.

   (b) Verify the area under the PDF is indeed unity.

   (c) What is the probability that the customer will have to wait at least 5 minutes?

4. Consider the function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 1, & x \geq \frac{1}{2}. \end{cases}$$

   (a) Sketch $F(x)$ and show that $F(x)$ satisfies the properties of a cdf.

   (b) If X is the random variable whose cdf is given by $F(x)$, find

      i. $P(X \leq 1/4)$,

      ii. $P(0 < X \leq 1/4)$.

Homework 4 solution

Problem 1

a) Here, the range of Y $\mathcal{R}_Y = \{-2, -1, 1, 2\}$. By the definition of pmf we have $\sum_{y \in \mathcal{R}_Y} p_Y(Y = y) = 1$.

$$\begin{aligned}
&\sum_{y \in \mathcal{R}_Y} p_Y(Y = y) = 1, \\
\Rightarrow\ & p_Y(Y = -2) + p_Y(Y = -1) + p_Y(Y = 1) + p_Y(Y = 2) = 1, \\
\Rightarrow\ & \frac{c}{2} + \frac{c}{1} + \frac{c}{1} + \frac{c}{4} = 1, \\
\Rightarrow\ & \frac{11c}{4} = 1, \\
\Rightarrow\ & c = \frac{4}{11}.
\end{aligned}$$

b) i) Then, we have

$$p_Y(Y = -2) = \frac{\frac{4}{11}}{(-2)^2} = \frac{1}{11}.$$

ii) Probability of $Y < 1$ is given by

$$\begin{aligned}
p_Y(Y < 1) =& p_Y(Y = -2) + p_Y(Y = -1) \\
=& \frac{\frac{4}{11}}{(-2)^2} + \frac{\frac{4}{11}}{(-1)^2} = \left(\frac{4}{11}\right)\left(\frac{5}{4}\right) = \frac{5}{11}.
\end{aligned}$$

Problem 2

a) According to Uniform distribution, if $X \sim U(a, b)$,

$$f_X(u) = \frac{1}{b - a}, \text{ for } a < u < b.$$

Then, the PDF of the resistance R is as follows.

$$f_R(r) = \begin{cases} \frac{1}{1150-850} = \frac{1}{300}, & \text{for } 850 \le u \le 1150. \\ 0, & \text{otherwise.} \end{cases}$$

b) Using the property of PDF we get

$$P(900\Omega \le R \le 950\Omega) = \int_{900}^{950} \frac{1}{300} dr = \frac{50}{300} = \frac{1}{6}.$$

Problem 3

a) As $f_X(x) = \frac{dF_X(x)}{dx}$, we have

$$f_X(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & 1 \le x \le 8, \\ \frac{1}{4}, & 8 \le x \le 10, \\ 0, & \text{otherwise.} \end{cases}$$

b) We have

$$P(0 < X \le 1/4) = F(1/4) - F(0) = 3/4 - 1/2 = 1/4.$$

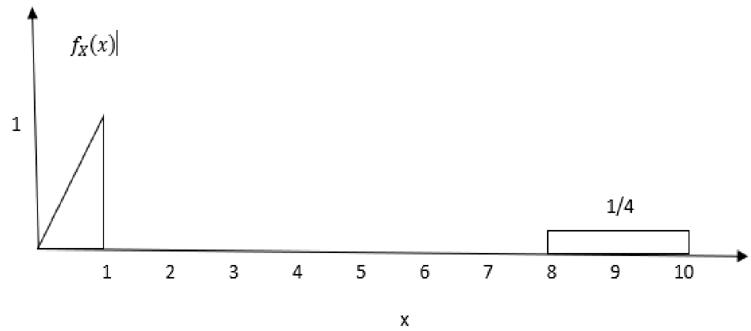

Figure 1: $f_X(x)$

c) The area under the PDF is as follows.

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^1 x dx + \int_8^{10} (\frac{1}{4})dx = \frac{1}{2} + \frac{1}{2} = 1$$

d) The probability that the customer will have to wait at least 5 minutes

$$p(X \geq 5) = \int_5^{10} f_X(x)dx = \int_8^{10} \frac{1}{4}dx = \frac{1}{2}.$$

Problem 4

a) The sketch of the given function is as follows.

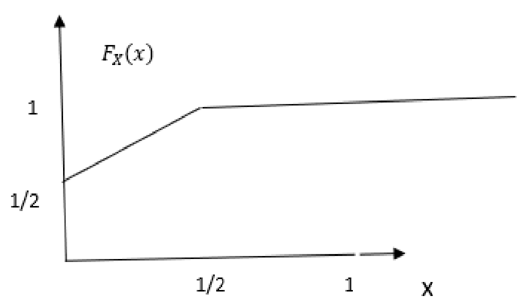

Figure 2: $F_X(x)$

From the figure, we can see that $0 \leq F(x) \leq 1$, and $F(x)$ is a non-decreasing function. Moreover, we have $F(-\infty) = 0$, $F(\infty) = 1$, and $F(x)$ is continuous on the right. Thus, $F(x)$ satisfies all the properties of a cdf.

b) i) We have

$$P(X \leq 1/4) = F(1/4) = 1/4 + 1/2 = 3/4.$$

ii) The probability is as follows.

$$P(0 < X \leq 1/4) = F(1/4) - F(0) = 1/4 + 1/2 - 1/2 = 1/4.$$