ECE 353 : Probability and Random Signals
Homework 8
Spring 2019

Due May 30, 2019

1. Consider two random variables X and Y that follows the joint PDF:

$$f_{XY}(x,y) = \begin{cases} c, & x+y<5,\ x\geq 0,\ y\geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   (a) Find the value of c.

   (b) Prove that X and Y are not independent.

2. Let $\{X(t) : t \geq 0\}$ be a Poisson process *i.e.*, $P(X(t) = n) = \frac{\{\lambda t\}^n exp(-\lambda t)}{n!}$. For $s = t/5$, show that the conditional distribution of $X(s)$ given that $X(t) = n$ is binomial with parameters n and $p = 1/5$, *i.e.*,

$$P(X(t/5) = m|X(t) = n) = \binom{n}{m}(1-p)^{n-m}p^m.$$

3. The joint PDF of X,Y is as follows.

$$f_{XY}(x,y) = \begin{cases} ce^{-x}e^{-y}, & x\geq 0, y\geq 0. \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

   (a) Find the value of c.

   (b) Find $f_Y(y)$.

   (c) Find $f_{X|Y}(x|y)$.

Homework 8 solution

Problem 1

1. We know that the joint PDF should satisfy

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{XY}(x,y)dxdy = 1\\
\Rightarrow &\int_{0}^{5}\int_{0}^{5-y} cdxdy = 1\\
\Rightarrow &c\int_{0}^{5}(5-y)dy = 1\\
\Rightarrow &c\big[5y - y^2/2\big]_0^5 = 1\\
\Rightarrow &c\big[25 - 25/2\big] = 1\\
\Rightarrow &25c/2 = 1\\
\Rightarrow &c = 2/25.
\end{aligned}$$

2. By definition of the marginal $f_X(x) : f_X(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)dy$. Substituting $f_{xy}(x,y)$ into the integral, we obtain

$$\begin{aligned}
f_X(x) = &\int_{-\infty}^{\infty} 2/25dy \qquad (3)\\
=&2/25\int_{0}^{5-x} dy = 2/25(5-x), \quad 0 \le x \le 1.
\end{aligned}$$

By definition of the marginal $f_Y(y) : f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$. Substituting $f_{XY}(x,y)$ into the integral, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} 2/25dx = 2/25\int_{0}^{5-y} dx = 2/25(5-y), \quad 0 \le y \le 1. \qquad (4)$$

Then, we have

$$f_X(x)f_Y(y) = 2/25(5-x)2/25(5-y) = 4/625(5-x)(5-y) \neq f_{XY}(x,y).$$

Therefore, X,Y are not independent.

Problem 2

$$\begin{aligned}
P(X(s) = m|X(t) = n) &= P(X(t/5) = m|X(t) = n)\\
&= \frac{P(X(t/5) = m, X(t) = n)}{P(X(t) = n)}\\
&= \frac{P(X(t) = n|X(t/5) = m)P(X(t/5) = m))}{P(X(t) = n)}\\
&= \frac{P(X(t) - X(t/5) = n - m)P(X(t/5) = m))}{P(X(t) = n)}\\
&= \frac{P(X(t - t/5) = n - m)P(X(t/5) = m))}{P(X(t) = n)}
\end{aligned}$$

As $X(t)$ has Poisson distribution,

$$\begin{aligned}
P(X(s) &= m|X(t) = n) \\
&= \frac{P(X(t-t/5) = n-m)P(X(t/5) = m))}{P(X(t) = n)} \\
&= \frac{\frac{\{\lambda(t-t/5)\}^{n-m} exp(-\lambda(t-t/5))}{n-m!} \frac{\{\lambda(t/5)\}^{m} exp(-\lambda(t/5))}{m!}}{\frac{\{\lambda(t)\}^{n} exp(-\lambda(t))}{n!}} \\
&= \frac{n!}{(n-m)!m!} \frac{\{\lambda(t-t/5)\}^{n-m}\{\lambda(t/5)\}^{m}}{\{\lambda(t)\}^{n}} \frac{exp(-\lambda(t-t/5))exp(-\lambda(t/5))}{exp(-\lambda(t))} \\
&= \tbinom{n}{m} \frac{t^{n-m}(1-1/5)^{n-m}(t^m 5^{-m}}{t^n} \\
&= \tbinom{n}{m}(1-1/5)^{n-m}(1/5)^m \\
&= \tbinom{n}{m}(1-p)^{n-m}p^m, \quad p = 1/5
\end{aligned}$$

So the conditional distribution of $X(s)$ given that $X(t) = n$ is binomial with parameters n and $p = 1/5$.

Problem 3

a)

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1 \\
\Rightarrow &\int_{0}^{\infty}\int_{0}^{x} ce^{-x}e^{-y}dydx = 1, \\
\Rightarrow &c\int_{0}^{\infty} e^{-x}\{\int_{0}^{\infty} e^{-y}dy\}dx = 1, \\
\Rightarrow &c\int_{0}^{\infty} e^{-x}[1-e^{-\infty}]dx = 1, \\
\Rightarrow &c\int_{0}^{\infty} e^{-x}dx = 1, \\
\Rightarrow &c[-e^{-x}]_0^{\infty} = 1, \\
\Rightarrow &c = 1.
\end{aligned}$$

b) We know

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\
&= e^{-y}\int_{0}^{\infty} e^{-x}dx \\
&= \begin{cases} e^{-y}, & y \geq 0, \\ 0, & \text{o.w.} \end{cases}
\end{aligned}$$

c) By definition, the conditional PDF is as follows.

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{5}$$

Substituting $f_{XY}(x,y)$ and $f_Y(y)$ in (5), we get

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-x}e^{-y}}{e^{-y}} = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{o.w.} \end{cases}$$