ECE 353 : Probability and Random Signals
Homework 10
Spring 2019

Due June 13, 2019

1. For an arbitrary constant a, let $Y(t) = X(t + 2a)$. If $X(t)$ is a wide sense stationary random process, is $Y(t)$ wide sense stationary?

2. Let W be an exponential random variable with pdf

$$f_W(w) = \begin{cases} e^{-w}, & w \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

   First order CDF of $X(t)$, if $X(t) = t - w$.

3. Consider the random sequence $X[n] = \sum_{m=1}^{n} \alpha^{n-m} W[m]$, for $n \geq 1$, with $|\alpha| < 1$. $W[n]$ is a Bernoulli random sequence, that is,

$$W[n] = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } q = 1 - p. \end{cases}$$

   Find $E\{X[n]\}$.

4. Consider the random process $X(t) = A\cos(\omega_0 t + \theta)$, where A and θ are independent, real-valued random variables. A has the mean μ_A and variance σ_A^2 and θ is uniformly distributed between -π and π. Calculate $E\{X[n]\}$ and $R_{XX}(t_1, t_0)$

Homework 10 solution

Problem 1

As $X(t)$ is wide sense stationary, we have

$$E[X(t)] \text{ is constant } \ \forall t, \tag{1}$$
$$R_X(t_1, t_2) \text{ is a function of } t_1 - t_2, \ \forall t_1, t_2. \tag{2}$$

Therefore, $Y(t)$ is also wide sense stationary.

Then, we have

$$\begin{aligned} E[Y(t)] &= E[X(t+2a)] = constant, \ \forall t. \\ R_Y(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\ &= E[X(t_1+2a)X(t_2+2a)] = \text{a function of } ((t_1+2a) - (t_2+2a)) = t_1 - t_2. \end{aligned}$$

Problem 2

First order CDF of X is as follows.

$$\begin{aligned} F_X(x,t) =& P(X(t) \le x) \\ =& P(t - w \le x), \ X(t) = t - w \\ =& P(w \ge t - x) \\ =& F_W(t-x) = \begin{cases} 1, \ x \ge t \\ \int_{t-x}^{\infty} e^{-w} dw = e^{-(t-x)}, \ t < x. \end{cases} \end{aligned}$$

Problem 3

$$\begin{aligned} E[X[n] =& E[\sum_{m=1}^{n} \alpha^{n-m} W[m]] \\ =& \sum_{m=1}^{n} \alpha^{n-m} E[W[m]] \\ =& \sum_{m=1}^{n} \alpha^{n-m} .p \\ =& p \sum_{m=1}^{n} \alpha^{n-m} \\ =& p \sum_{k=0}^{n-1} \alpha^{k} \\ =& p \frac{1-\alpha^n}{1-\alpha} \end{aligned}$$

Problem 4

1.

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\begin{aligned}
E\{X[n]\} &= E[A\cos(\omega_0 t + \theta)] \\
&= E[A]E[\sin(\omega_0 t + \theta)], \quad \text{As } A \text{ and } \theta \text{ are independent} \\
&= \mu_A \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(\omega_0 t + \theta) d\theta, \quad \theta \text{ is uniform random variable} \\
&= \frac{\mu_A}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) d\theta \\
&= \frac{\mu_A}{2\pi} .0 \\
&= 0.
\end{aligned}
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2.

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\begin{aligned}
R_{XX}(t_1, t_0) &= E[X(t_1)X^*(t_0)] \\
&= E[A\cos(\omega_0 t_1 + \theta)A\cos(\omega_0 t_0 + \theta)] \\
&= E[A^2]E[\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_0 + \theta)] \\
&= \frac{\sigma_A^2 + \mu_A^2}{2} E[\cos(\omega_0 t_0 + \omega_0 t_1 + 2\theta) + \cos(\omega_0 t_0 - \omega_0 t_1)] \\
&= \frac{\sigma_A^2 + \mu_A^2}{2} [E[\cos(\omega_0 t_0 + \omega_0 t_1 + 2\theta)] + E[\cos(\omega_0 t_0 - \omega_0 t_1)]] \\
&= \frac{\sigma_A^2 + \mu_A^2}{2} [0 + \cos(\omega_0 t_0 - \omega_0 t_1)] \\
&= \frac{\sigma_A^2 + \mu_A^2}{2} \cos(\omega_0 t_0 - \omega_0 t_1)
\end{aligned}
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