1 (a) (i) $S = \{ \text{'S'}, \text{'FS'}, \text{'FFS'}, \text{'FFF'} \}$

'S' – single successful transmission

'FS' – an unsuccessful transmission followed by a successful transmission.

⋮

'FFF' – 3 unsuccessful trans.

$A = \{ \text{'FS'} \}$ (since the number of letters indicates the number of trans., 'FS' is the only outcome with an even no. of trans.)

$B = \{ \text{'FFF'} \}$ (3 failed trans for the event of no acknowledgement)

(ii) Show $A \cap B = \phi$ to establish M.E

since $\underset{\{\text{'FS'}\}}{A} \cap \underset{\{\text{'FFF'}\}}{B} = \phi \rightarrow$ A and B are M.E.

(iii) Show $P(A \cap B) = P(A) \cdot P(B)$ to establish indep.

$A \cap B = \phi \rightarrow P(A \cap B) = 0$

$P(A) = P(\{\text{'FS'}\}) = \underbrace{0.2}_{\text{first fail}} \cdot \underbrace{0.8}_{\text{2nd success}}$

$P(B) = P(\{\text{'FFF'}\}) = \underbrace{0.2}_{\text{1st F}} \cdot \underbrace{0.2}_{\text{2nd F}} \cdot \underbrace{0.2}_{\text{3rd F}} = 0.2^3$

$0 \neq (0.2 \cdot 0.8) \cdot (0.2)^3$

$\Rightarrow P(A \cap B) \neq P(A) \cdot P(B)$

$\Rightarrow$ A & B are not indep.

1) b) $S = \{1,2,3,4,5,6,7,8\}$ $\left(P(\{'1'\}) = P(\{'2'\}) = \ldots = P(\{'8'\}) = \frac{1}{8} \right)$

(i) $A = \{1,2\}$ (1st team less ques. than $\underset{3Q}{\overset{\shortparallel}{\text{2nd team}}}$)

$P(A) = \frac{2}{8}$

(ii) $B = \{4,5,6,7,8\}$ ($\underset{3Q}{\text{2nd team}}$ less ques. than 1st team)

$P(B) = \frac{5}{8}$

(iii) $C = \{2,4,6,8\}$ (even no. of ques.)

$$P(C|B) \underset{\substack{\uparrow \\ \text{By cond.} \\ \text{def.}}}{=} \frac{P(C \cap B)}{P(B)} = \frac{P(\overset{C\cap B}{\overset{\shortparallel}{\{4,6,8\}}})}{P(B)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}.$$

1) c) $P(\text{'D'}) = 0.8$

3

(i) A = 'At least one detection (D) in 3 $\underbrace{\text{tries}}_{\text{indep.}}$'

$\bar{A}$ = 'no D in 3 tries'

$\boxed{P(A)} = 1 - P(\bar{A}) = 1 - (1-0.8)^3 = \boxed{1 - 0.2^3}$ $(= 0.992)$

Alternative,

A = 'exactly 1D' or 'exactly 2D' or 'exactly 3D'

$P(A) = P(\text{'1D'} \cup \text{'2D'} \cup \text{'3D'}) \overset{\text{By M.E.}}{=} P(\text{'1D'}) + P(\text{'2D'}) + P(\text{'3D'})$

$= \binom{3}{1} 0.8^1 \cdot 0.2^2 + \binom{3}{2} \cdot 0.8^2 \cdot 0.2 + \binom{3}{3} 0.8^3 \cdot 0.2^0$

$= 3 \cdot 0.8 \cdot 0.2^2 + 3 \cdot 0.8^2 \cdot 0.2 + 0.8^3$ $(= 0.992)$

(ii) B = '3 or more D's in 5 tries'

B = '3 D's' OR '4 D's' OR '5 D's'

$P(B) = P(\{3D\} \cup \{4D\} \cup \{5D\})$

$= P(\{3D\}) + P(\{4D\}) + P(\{5D\})$

$= \binom{5}{3} 0.8^3 \cdot 0.2^2 + \binom{5}{4} 0.8^4 \cdot 0.2 + \binom{5}{5} 0.8^5 \cdot (0.2)^0$

$= \boxed{10 \cdot 0.8^3 \cdot 0.2^2 + 5 \cdot 0.8^4 \cdot 0.2 + 0.8^5}$ $(= 0.94208)$

$\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2!} = \frac{20}{2} = 10.$

$\binom{5}{4} = \frac{5!}{4!1!} = \frac{5}{1} = 5.$

1(c) (iii) $C =$ '5 D's' (five detections in 5 tries)

$C = \{5D\}$ $\left(C \cap B = \{5D\} \cap (\{3D\} \cup \{4D\} \cup \{5D\}) = \{5D\} = C \right)$

$$P(C \mid B) = \frac{P(C \cap B)}{P(B)} = \frac{P(C)}{P(B)} = \frac{P(\{5D\})}{P(B)} = \frac{0.8^5}{10 \cdot 0.8^3 \cdot 0.2 + 5 \cdot 0.8^4 \cdot 0.2 + 0.8^5}$$

(iv) $E = \{$ at least 1D in n tries $\}$

$\bar{E} = \{$ no detect. in n tries $\}$

We want: $P(E) \geq 0.99$

$$P(E) = 1 - P(\bar{E}) = 1 - 0.2^n$$

$$\Rightarrow \boxed{1 - 0.2^n \geq 0.99}$$

$0.2^n \leq 0.01 \rightarrow n \log(0.2) \leq \log(0.01)$ $\div$ by $\log(0.2) < 0$

$$\boxed{n \geq \frac{\log(0.01)}{\log(0.2)}} = 2.86\ldots$$

$\Rightarrow n = 3.$

1) d)

student id	1	2	...	20
project	0/1	0/1		0/1

i) number of assigment with 10 students in each project

number of 20-digit binary numbers with $\underset{\text{ten}}{10}$ '0's and $\underset{\text{ten}}{10}$ '1's.

$\Leftrightarrow$ number of 10 element subsets of $\{1, 2, \ldots, 20\}$ → combinations (indicating, for example, the students in proj. 1)

$$\boxed{\binom{20}{10}}$$

ii) Consider a 'bag' containing 10 'C' and 10 'E'.

Draw (w/o replacements) 20 letters. The first 10 go to proj. 1
without

The next 10 " " proj 2.

$A = \{ '\underbrace{C \cdots C}_{10}\underbrace{E \cdots E}_{10}', '\underbrace{E \cdots E}_{10}\underbrace{C \cdots C}_{10}' \}$ = Cs only / Es only event

Total no. of sequences with 10'C' and 10'E' $= \binom{20}{10}$

Total no. of sequences in $A = 2$

$$\Rightarrow \frac{2}{\binom{20}{10}}$$

iii) B = first 10 letters drawn have 5 'C's and 5 'E's
next 10 " " " " " "

Total n. of sequence in B $\binom{10}{5}\cdot\binom{10}{5}$

$$\Rightarrow \text{Prob} = \frac{\binom{10}{5}\binom{10}{5}}{\binom{20}{10}}$$

B :

$$\underbrace{\overbrace{E\,C\,E\,E\,C\,E\,C\,C\,E\,C}^{5E\,5C}}_{\binom{10}{5}\,\text{seq}} \times \underbrace{\overbrace{\qquad\qquad\qquad}^{5E\,5C}}_{\binom{10}{5}\,\text{seq}}$$

$\Rightarrow \binom{10}{5}\cdot\binom{10}{5}$ seq.
in which first 10 contain 5E/5C
and next 10 " 5E/5C

2)a) i) Prove that $0 \le F_X(x) \le 1$:

Pf: By def: $F_X(x) = P(\{X \le x\})$

since prob. are always non-negative (axiom 1) $\cancel{P(A) \ge 0}$ $P(A) \ge 0$ for any $A \subseteq S$.

and prob. are always less than one by one of the properties derived in class $P(A) \le 1$ for any $A \subseteq S$,

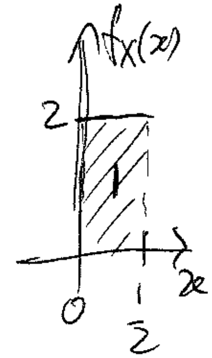
we have $0 \le P(\{X \le x\}) \le 1$

$\Rightarrow \quad 0 \le F_X(x) \le 1.$ ∎

Can $f_X(x)$ (PDF of X) > 1 ?

Yes, by the following example:

$$f_X(x) = \begin{cases} 2, & 0 < x \le \frac{1}{2} \\ 0, & \text{o.w} \end{cases}$$



check that $f_X(x)$ is PDF

① $f_X(x) \ge 0$ (all values of $f \ge 0$) $2 \ge 0$, $0 \ge 0$

② $\int_{-\infty}^{\infty} f_X(x) = 1$ Area $= 2 \cdot \frac{1}{2} = 1$.

$f_X(x) > 1$ for $0 \le x \le 2$.

2) b) i)

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 & x < 6 \\ \frac{1}{4} & 6 < x < 10 \\ 0 & x > 10 \end{cases}$$

$$= \begin{cases} \frac{1}{4}, & 6 < x < 10 \\ 0, & \text{o.w} \end{cases}$$

Note that the PDF is undefined at 6 and 10. However, assigning a finite value to $f_X(6)$ and $f_X(10)$ will not affect any probability calculation.

(ii) $P(\{X<7\} \cup \{X>9\}) = P(\{X<7\}) + P(\{X>9\})$

$= P(\{X<7\}) + 1 - P(\{X \le 9\})$

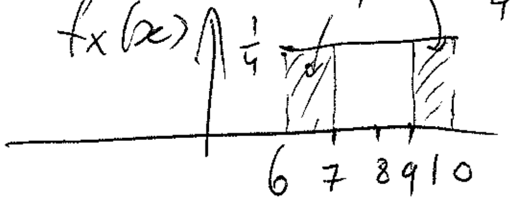
$= F_X(7^-) + 1 - F_X(9)$

$= \frac{7-6}{4} + 1 - \left(\frac{9-6}{4}\right)$

$= \frac{1}{4} + \underbrace{1 - \frac{3}{4}}_{\frac{1}{4}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

Or

$1 \times \frac{1}{4} + 1 \times \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$



2) b) iii)

$$P(\{X=8\}) = P(\{8 \le X \le 8\})$$

$$= F_X(8) - F_X(8^-) \qquad \left(\begin{array}{l} P(a \le X \le b) \\ = F_X(b) - F_X(a^-) \end{array} \right)$$

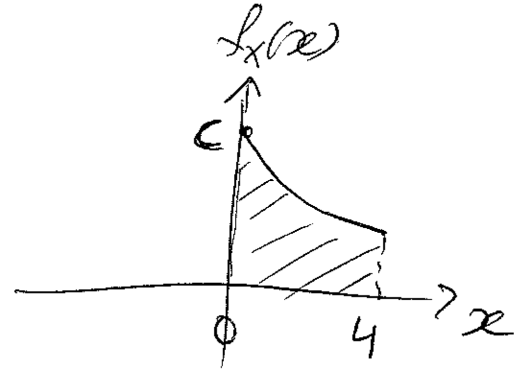
$$= \frac{8-6}{4} - \left(\frac{8-6}{4}\right)$$

Note: can replace 8^- with 8 by the continuity of $F_X(x)$

$$= \frac{2}{4} - \frac{2}{4} = 0.$$

Also: The probability of a singleton even (i.e., a single value of the RV) is zero for continuous RVs.

2 c $f_X(x) = \begin{cases} c e^{-2x}, & 0 \le x \le 4 \\ 0, & \text{o.w} \end{cases}$



i Find c! $\underbrace{\int_{-\infty}^{\infty} f_X(x)}_{} = 1$

$$LHS = \int_0^4 c e^{-2x} dx = c \int_0^4 e^{-2x} dx = c \left. \frac{e^{-2x}}{(-2)} \right|_0^4 = \frac{c}{(-2)}(e^{-8} - 1)$$

$$= \frac{c}{2}(1 - e^{-8})$$

$$LHS = 1 \Rightarrow \frac{c}{2}(1 - e^{-8}) = 1 \Rightarrow$$

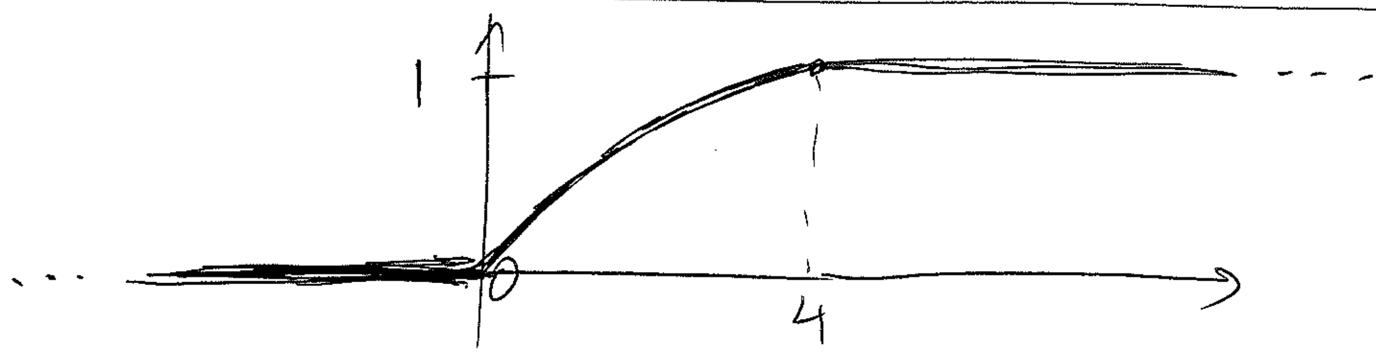
$$\boxed{c = \frac{2}{1 - e^{-8}}}$$

ii Find $F_X(x)$, sketch:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ c \int_0^x e^{-2t} dt & 0 \le x \le 4 \\ 1 & x > 4 \end{cases} = \begin{cases} 0 & x < 0 \\ c \left. \frac{e^{-2t}}{(-2)} \right|_0^x & 0 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ c \cdot \frac{(1-e^{-2x})}{2} & 0 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

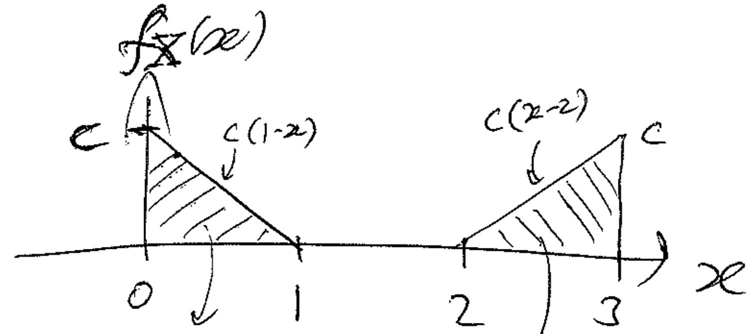
$$\boxed{F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1-e^{-2x}}{1-e^{-8}} & 0 \le x \le 4 \\ 1 & x > 4 \end{cases}}$$



(iii)

$$P(\{1 \le X \le 2\}) = F_X(2) - F_X(1^-)$$
$$= F_X(2) - F_X(1) \quad \text{by continuity of CDF}$$
$$= \frac{1-e^{-4}}{1-e^{-8}} - \frac{1-e^{-2}}{1-e^{-8}} = \frac{e^{-2}-e^{-4}}{1-e^{-8}} = \boxed{\frac{e^{-2}(1-e^{-2})}{(1-e^{-8})}}$$

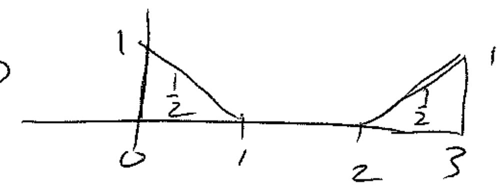
2) d) i)



$$\int_{-\infty}^{\infty} f_X(x)\,dx = \underbrace{\frac{1\cdot c}{2}}_{\int_0^1 f_X(x)dx} + \underbrace{\frac{1\cdot c}{2}}_{\int_1^2 f_X(x)dx} = \frac{c}{2}\cdot 2 = c. \quad = 1 \Rightarrow \boxed{c=1}.$$

(Graphically).

$\Rightarrow$



ii) CDF:

$$F_X(x) = \begin{cases} 0 & x<0 \\ \int_0^x (1-t)\,dt & 0 \le x < 1 \\ \underbrace{\int_0^1 (1-t)\,dt}_{\frac{1}{2}} + \boxed{\int_1^x f_X(t)\,dt} = 0 & 1 \le x < 2 \\ \underbrace{\int_0^1 (1-t)\,dt}_{\frac{1}{2}} + \overbrace{\int_1^2 f_X(x)\,dx}^{0} + \int_2^x f_X(t)\,dt \;\; (t-2)dt & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

2 d (ii) (cont'd) $\boxed{13}$

$F_X(x)$:

- $\int_0^x (1-t)\,dt = -\frac{(1-t)^2}{2}\Big|_0^x = \frac{1-(1-x)^2}{2} = \frac{2x-x^2}{2}$
- $\int_2^x (t-2)\,dt = \frac{(t-2)^2}{2}\Big|_2^x = \frac{(x-2)^2}{2}$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{2x-x^2}{2}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{1}{2} + \frac{(x-2)^2}{2}, & 2 \le x < 3 \quad \text{(only for } x=2) \\ 1, & x > 3 \end{cases}$$

2 d (iii)

$P(X \le x) = P(X > x)$

$\Rightarrow F_X(x) = 1 - F_X(x) \Rightarrow 2F_X(x) = 1$

$\Rightarrow F_X(x) = \frac{1}{2}.$

The values of x for which $F_X(x) = \frac{1}{2}$ are $\boxed{1 \le X \le 2}$.

(iv) $P(\{X = 0.5\}) = \cancel{P(0.5 \le X \le 0.5)}$ $P(\{0.5 \le X \le 0.5\})$ $= F_X(0.5) - F_X(0.5^-) \overset{\text{by cont. of } F_X(x) @ 0.5}{=} \frac{3}{8} - \frac{3}{8} = 0.$