1) a) c)

6

S
F S
F F S
F F F

$S = \{ 's', 'fs', 'ffs', 'fff' \}$

Turn 1 2 3 3

$A = \{ 'fs' \}$

$B = \{ 'fff' \}$

4 (ii) $\boxed{A \cap B = \emptyset}$ YES, A & B are M.E.

4 (iii)

$P(A \cap B) = P(\emptyset) = 0$

$P(A) > 0$, $P(B) > 0$

$\left(\begin{array}{l} P(A) = 0.2 \cdot 0.8 > 0 \\ P(B) = (0.2)^3 > 0 \end{array} \right)$

$P(A) \cdot P(B) > 0$

$\Rightarrow \boxed{P(A \cap B) \neq P(A) \cdot P(B)}$

$S = \{1, 2, \dots, 8\}$

1 (b) (i) $A =$ "First team wins" $= \{1, 2\}$

5

$P(A) = P(\{1,2\}) = \frac{2}{8}$

5 (ii) $B =$ "Second team wins" $= \{4, 5, 6, 7, 8\}$

$P(B) = P(\{4,5,6,7,8\}) = \frac{5}{8}$

7 (iii) $C =$ "number is even" $= \{2, 4, 6, 8\}$

$P(C \mid B) \overset{\text{conditional def}}{=} \frac{P(C \cap B)}{P(B)} = \frac{P(\overbrace{\{4,6,8\}}^{C \cap B})}{P(B)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}.$

1(c) (iii) C = " ~~detect~~ D in all 5 "

5

$$P(C \mid B) = \frac{P(C \cap B)}{P(B)} = \frac{P(C)}{P(B)} = \frac{\binom{5}{5} 0.8^5 \cdot 0.2^0}{\binom{5}{3} 0.8^3 0.2^2 + \binom{5}{4} 0.8^4 0.2 + \binom{5}{5} 0.8^5}$$

5 (iv) B = { at least 1 D in n tries }

$\bar{B}$ = { no D in n tries }

W.T. determine n s.t

$P(B) \geq 0.99$

$$P(B) = 1 - P(\bar{B}) = 1 - 0.2^n$$

$$\Rightarrow \quad 1 - 0.2^n \geq 0.99$$

$$0.01 \geq 0.2^n$$

$$\log(0.01) \geq n \underbrace{\log(0.2)}_{<0} \rightarrow \boxed{\frac{\log(0.01)}{\log(0.2)} \leq n}$$

1 (c) $P(\text{'}\overset{\text{detect}}{D}\text{'}) = 0.8$

4 (c) $A =$ "at least 1D in 3 tries"

$P(A) = 1 - P(\bar{A}) = 1 - P(\text{"no det. in 3 tries"})$

$= 1 - (1-0.8)^3$ ✓

$= 1 - 0.2^3$ (or $\binom{3}{1}0.8^1 \cdot 0.2^2 + \binom{3}{2}0.8^2 \cdot 0.2^1 + \binom{3}{3} \cdot 0.8^3$)

5 (ii) $B =$ "3 or more Ds in 5 tries"

$B = \{\text{exactly } 3\} \cup \{\text{exactly } 4\} \cup \{\text{exactly } 5\}$

$P(B) = P(\{3\}) + P(\{4\}) + P(\{5\})$

$= \binom{5}{3} 0.8^3 \cdot \underset{(1-0.8)}{0.2^2} + \binom{5}{4} 0.8^4 \cdot 0.2 + \binom{5}{5} 0.8^5 \underset{1}{\cancel{0.2^0}}$

$\frac{25}{3}$ / 2

1 d) (i)

7

student $1, 2, \ldots 20$

proj $0/1$ $0/1$ $0/1$

$\Rightarrow$ number of assignments

= the number of binary seq. with 10 '0' and 10 '1's

= number of 10 element subsets of $\{1, \ldots 20\}$

$$= \binom{20}{10} = \frac{20!}{10!\,10!}$$

$\boxed{ECECE \ldots}$ S = all seq of the type with 10 E, 10 C

(ii)

~~EE EEE~~ E only / C only = $\left\{ \begin{array}{l} E \cdots E\,C \cdots C \\ \text{or } C\,C \cdots C\,C \cdots \end{array} \right\}$

$$\Rightarrow \frac{2}{\binom{20}{10}}$$

1 d iii

7

$S =$ all seq with $10E$, $10C$

$A =$ $5E, 5C$ on the first 10 places

$\underbrace{ECEECECECCY}_{5/5} \quad 5/5$

$$\frac{\binom{10}{5}\binom{10}{5}}{\binom{20}{10}} = \frac{\frac{10!}{5!5!} \cdot \frac{10!}{5!5!}}{\frac{20!}{10!10!}} = \frac{10!\,10!\,10!\,10!}{5!\cdot 5!\cdot 5!\cdot 5!\ 20!}$$

$$= \frac{(10!)^4}{(5!)^4\ 20!}$$

2) a) $\boxed{0 \le f_X \le 1}$

10

$\Rightarrow$ Proof: By def $F_X(x) = P(\{X \le x\})$

By axiom 1: $P(A) \ge 0 \Rightarrow P(\{X \le x\}) \ge 0$

$\Rightarrow F_X(x) \ge 0$

By property $(P(A) \le 1) \rightarrow P(\{X \le x\}) \le 1$

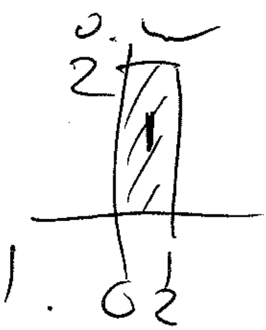
$\rightarrow F_X(x) \le 1$

$\rightarrow \boxed{0 \le F_X(x) \le 1}$ ✓

4

Can $f_X(x) > 1$?

Yes, example $f_X(x) = \begin{cases} 2, & 0 < X \le \frac{1}{2} \\ 0, & \text{o.w.} \end{cases}$

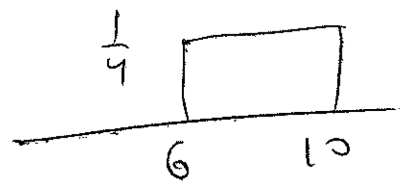


$f_X(x)$ is valid pdf $\begin{cases} f_X(x) \ge 0 \\ \int f_X(x)dx = 1. \end{cases}$

Note $f_X(x) > 1$ for $0 \le x \le \frac{1}{2}$.

2) (b) 6

(i) $f_X(x) = \dfrac{dF_X(x)}{dx} = \begin{cases} 0 & x < 6 \\ \frac{1}{4} & 6 < x < 10 \\ 0 & x > 10 \end{cases}$



(ii) $P(\{X < 7\} \cup \{X > 9\})$

7

$= P(\{X < 7\}) + P(\{X > 9\})$

$= P(\{X < 7\}) + 1 - P(\{X \le 9\})$

$= F_X(7) + 1 - F_X(9)$

$= \dfrac{7-6}{4} + \underbrace{1 - \underbrace{\left(\dfrac{9-6}{4}\right)}_{\frac{3}{4}}}_{\frac{1}{4}}$

$= \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$

iii) **4**

$$P(\{X=8\}) = P(\cancel{\{X \le 8\}}\ \{8 \le X \le 8\})$$

$$= F_X(8) - F_X(8^-)$$

$$= \left(\frac{8-6}{4}\right) - \left(\frac{8-6}{4}\right) = 0.$$

(cont prob. → prob. of a single outcome = 0.)

2 e

$$f_X(x) = \begin{cases} c e^{-2x}, & 0 \le x \le 4 \\ 0, & \text{o.w} \end{cases}$$

i

6

$$1 = \int_{-\infty}^{\infty} f_X(x) \Rightarrow \int_0^4 c e^{-2x} dx = c \left. \frac{e^{-2x}}{(-2)} \right|_0^4$$

$$= c \, \frac{e^{-2 \cdot 4} - e^0}{(-2)}$$

$$= \frac{c}{2} \cdot (1 - e^{-8})$$

$$\Rightarrow \frac{c}{2}(1 - e^{-8}) = 1 \Rightarrow \boxed{c = \frac{2}{1 - e^{-8}}}$$

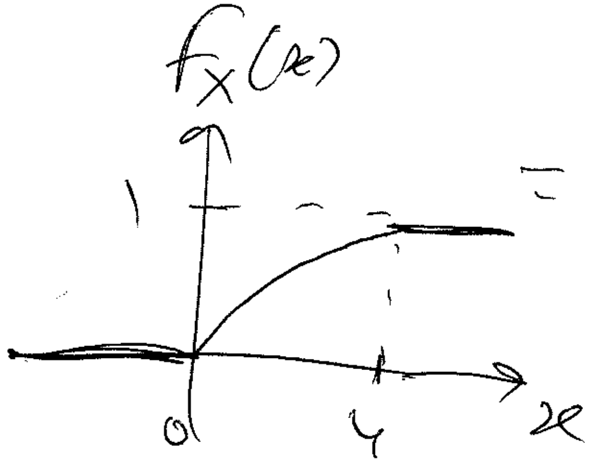
7 i CDF $F_X(x) = \int_{-\infty}^{x} f_X(t) dt$

(7 as on prev. page)

2) c) ii) (cont'd)

$$F_X(x) = \begin{cases} \int_{-\infty}^{x} 0\,dx = 0 & x < 0 \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{2}{1-e^{-8}} \cdot e^{-2t}\,dt & 0 \le x \le 4 \\ \int_{-\infty}^{0} 0\,dt + \underbrace{\int_{0}^{4} f_X(t)\,dt}_{1} + \underbrace{\int_{4}^{x} 0\,dt}_{0} & x > 4 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \frac{2}{1-e^{-8}} \left. \frac{e^{-2t}}{(-2)} \right|_0^x = \frac{\cancel{2}}{1-e^{-8}} \frac{(1-e^{2x})}{\cancel{2}} & 0 \le x \le 4 \\ 1 & x > 4 \end{cases}$$



$$= \begin{cases} 0 & x < 0 \\ \frac{1-e^{-2x}}{1-e^{-8}} & 0 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

2 (c) (iii)

6

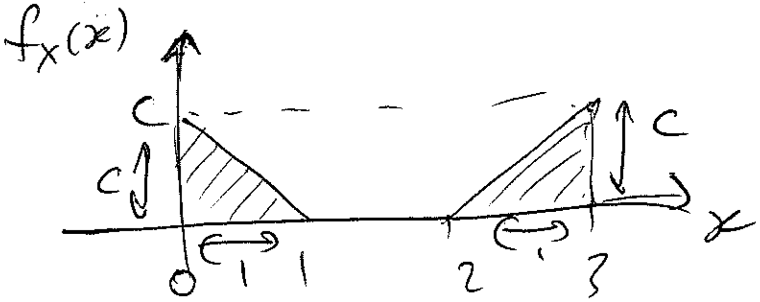
$$P(\{1 \le X \le 2\})$$

$$= F_X(2) - F_X(1^-)$$

$$= \frac{1-e^{-4}}{1-e^{-8}} - \frac{1-e^{-2}}{1-e^{-8}}$$

$$= \frac{e^{-2}-e^{-4}}{1-e^{-8}}.$$

2) a)



5) i) $\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{c \cdot 1}{2} + \frac{c \cdot 1}{2} = \frac{c}{2} + \frac{c}{2} = c = 1$

graphically

$\Rightarrow c = 1$

ii)

5)

$$f_X(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ 0 & 1 < x < 2 \\ x-2 & 2 \le x < 3 \\ 0 & \text{o.w.} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x (1-t)\,dt & 0 \le x \le 1 \\ \underbrace{\int_0^1 (1-t)\,dt}_{\frac{1}{2}} + \underbrace{\int_0^x 0\,dt}_{0} & 1 < x \le 2 \\ \frac{1}{2} + 0 + \int_2^x (t-2)\,dt & 2 \le x \le 3 \\ \frac{1}{2} + 0 \quad \underbrace{\int_2^3 (t-2)\,dt}_{\frac{1}{2}} + \underbrace{\int_3^x 0\,dt}_{0} = 1 & 3 < x \end{cases}$$

$-\frac{(1-t)^2}{2}$

$$
\begin{cases}
0 & x < 0 \\
-\frac{(1-t)^2}{2}\Big|_0^x = \frac{1-(1-x)^2}{2} = \left(\frac{2x-x^2}{2}\right) & 0 \le x < 1 \\
\frac{1}{2} & 1 \le x < 2 \\
\frac{1}{2} + \frac{(t-2)^2}{2}\Big|_2^x = \boxed{\frac{1}{2} + \frac{(x-2)^2}{2}} & 2 \le x < 3 \\
1 & x \ge 3
\end{cases}
$$

$$
= \begin{cases}
0 & x < 0 \\
\frac{2x-x^2}{2} & 0 \le x < 1 \\
\frac{1}{2} & 1 \le x < 2 \\
\frac{1}{2} + \frac{(x-2)^2}{2} & 2 \le x < 3 \\
1 & x \ge 3
\end{cases}
$$

(iii)

(5)

$$P(X \le x) = P(X > x)$$

$$F_X(x) = 1 - F_X(x) \Rightarrow \boxed{F_X(x) = \tfrac{1}{2}}$$

$$\Rightarrow \boxed{1 \le x \le 2} \Rightarrow F_X(x) = \tfrac{1}{2}$$

(iv)

(5)

$$P(\{X = 0.5\}) = 0$$

(continuous)

$$= F_X(0.5) - F_X(0.5^-)$$

$$= \frac{2 \cdot 0.5 - 0.5^2}{2} - \frac{2 \cdot 0.5 - 0.5^2}{2} = \frac{3}{8} - \frac{3}{8} = 0.$$