1 (a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$1 = \int_0^a cx\,dx = c\int_0^a x\,dx = c\,\frac{x^2}{2}\Big|_0^a = c\left(\frac{a^2-0^2}{2}\right) = \frac{ca^2}{2}$$

$$\rightarrow \boxed{c = \frac{2}{a^2}} \Rightarrow f_X(x) = \begin{cases} \frac{2}{a^2}x, & 0 \le x \le a \\ 0, & \text{o.w} \end{cases}$$

(b) $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^a x \cdot \frac{2}{a^2}x\,dx = \int_0^a \frac{2}{a^2}x^2\,dx$$

$$= \frac{2}{a^2}\,\frac{x^3}{3}\Big|_0^a = \frac{2}{a^2}\,\frac{a^3}{3} = \boxed{\frac{2}{3}a}$$

(c) $$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^a x^2 \cdot \frac{2}{a^2}x\,dx = \frac{2}{a^2}\int_0^a x^3\,dx = \frac{2}{a^2}\,\frac{x^4}{4}\Big|_0^a$$

$$= \frac{2}{a^2}\,\frac{a^4}{4} = \frac{2a^2}{4} = \boxed{\frac{a^2}{2}}$$

(d) $$Var(X) = E[X^2] - (E[X])^2 = \frac{a^2}{2} - \left(\frac{2}{3}a\right)^2 = a^2\left(\frac{1}{2} - \frac{4}{9}\right) = \boxed{\frac{a^2}{18}}$$

$\frac{9}{18} - \frac{8}{18}$

(2) $P_X(x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \\ 0, & \text{o.w} \end{cases}$

(a) $E[X] = \sum_{x \in \mathcal{X}} x \, P_X(x) = \sum_{x=0}^{1} x P_X(x) = 1 \cdot \overbrace{P_X(1)}^{p} + 0 \cdot \overbrace{P_X(0)}^{(1-p)}$

$= \boxed{p.}$

(b) $E[X^2] = \sum_{x \in X} x^2 P_X(x) = \sum_{x=0}^{1} x^2 P_X(x) = 1^2 \cdot p + 0 \cdot (1-p)$

$= \boxed{p.}$

(c) $Var(X) = E[X^2] - (E[X])^2 = p - p^2 = \boxed{p(1-p)}$

(3)

$X \sim Exp(\lambda) \rightarrow f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & o.w. \end{cases}$

$Y = \sqrt{X} \rightarrow$ mono. for $x \geq 0$

(a) $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$

$= \begin{cases} \lambda e^{-\lambda y^2} |2y|, & y \geq 0 \\ 0, & o.w. \end{cases}$

$g(X) = \sqrt{X} = Y$

$\rightarrow X = Y^2$

$\Rightarrow X = \boxed{g^{-1}(y) = y^2}$

$\boxed{\frac{dg^{-1}(y)}{dy} = 2y}$

$= \boxed{\begin{cases} 2\lambda y e^{-\lambda y^2}, & y \geq 0 \\ 0, & o.w. \end{cases}}$

(b) $E[Y^2] =$ easy way $E[X] = \frac{1}{\lambda}$ (from $E[X]$ for $X \sim exp(\lambda)$)

Derivation: $t = \lambda x$, $x = \frac{t}{\lambda}$

$= \int_0^\infty x \lambda e^{-\lambda x} dx = \cancel{\int_0^\infty y e^{-y}} = \int_0^\infty t e^{-t} \frac{dt}{\lambda} = \frac{1}{\lambda} \underbrace{\int_0^\infty t e^{-t} dt}_{1}$

4 a) $E[Y^n] = ?$

$$Y = aX \rightarrow E[(aX)^n] = E[a^n X^n]$$

$$= \boxed{a^n E[X^n]}.$$

b) $E[(Y - E[Y])^n] = ?$

$$E[Y] = a \underbrace{E[X]}_{0} = 0.$$

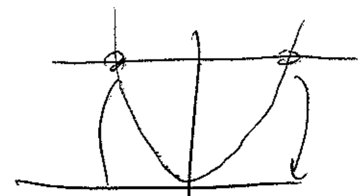
(from a)

$$\Rightarrow E[(Y - E[Y])^n] = E[(Y-0)^n] = E[Y^n] \underset{\text{from (a)}}{=} \boxed{a^n E(X^n)}.$$

Note since $E(Y)=0$, the answer to (a) and (b) are the same (moment = centered moment if $E[Y]=0$).

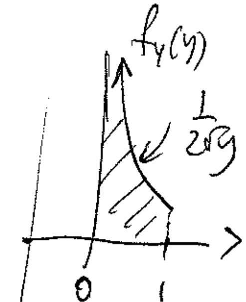
5 $X \sim U[-1,1]$ $\quad Y = X^2$.

a) $f_Y(y) = ?$




$$f_Y(y) = \begin{cases} f_X(g_1^{-1}(y))\left|\frac{dg_1^{-1}(y)}{dy}\right| + f_X(g_2^{-1}(y))\left|\frac{dg_2^{-1}(y)}{dy}\right| & y \geq 0 \\ 0 & y < 0 \end{cases}$$

check




$\int_0^1 \frac{1}{2\sqrt{y}}\,dy = \sqrt{y}\Big|_0^1 = 1-0 = 1$ ✓

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \leq X \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

$y \geq 0$ $\quad g_1^{-1}(y) = -\sqrt{y} \rightarrow \frac{dg_1^{-1}(y)}{dy} = -\frac{1}{2}\frac{1}{\sqrt{y}} \rightarrow \left|\frac{dg_1^{-1}(y)}{dy}\right| = \frac{1}{2}\frac{1}{\sqrt{y}}$

$g_2^{-1}(y) = \sqrt{y} \rightarrow \frac{dg_2^{-1}(y)}{dy} = +\frac{1}{2}\frac{1}{\sqrt{y}} \rightarrow \left|\frac{dg_2^{-1}(y)}{dy}\right| = \frac{1}{2}\frac{1}{\sqrt{y}}$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2}\cdot\frac{1}{2\sqrt{y}} + \frac{1}{2}\frac{1}{2\sqrt{y}} = \frac{1}{2}\frac{1}{\sqrt{y}} & \cancel{-}0 \leq \sqrt{y} \leq 1 \rightarrow 0 \leq y \leq 1 \\ 0, & \text{o.w} \end{cases}$$

5 b

one way

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 \frac{y}{2\sqrt{y}}\,dy$$

$$= \int_0^1 \frac{\sqrt{y}}{2}\,dy$$

$$= \frac{y^{\frac{3}{2}}}{\cancel{2}\cdot\frac{3}{\cancel{2}}}\Big|_0^1 = \frac{1}{3}.$$

another way

$$E[Y] = E[X^2]$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}\,dx = \frac{x^3}{2\cdot 3}\Big|_{-1}^{1} = \frac{x^3}{6}\Big|_{-1}^{1} = \frac{1^3-(-1)^3}{6} = \frac{2}{6} = \frac{1}{3}.$$

(6) $P_X(x) = \begin{cases} \frac{1}{2}, & x = \pm 1 \\ 0, & \text{o.w} \end{cases}$

(a) $h(s) = E[e^{sx}]$

$$= \sum_{x \in \mathcal{X}} e^{sx} P_X(x) = \frac{1}{2} e^{s \cdot 1} + \frac{1}{2} e^{s \cdot (-1)}$$

$$= \frac{e^{s} + e^{-s}}{2}.$$

(b) $$E[X^n] = \sum_{x} x^n P_X(x) = 1^n \cdot \frac{1}{2} + (-1)^n \cdot \frac{1}{2}$$

$$= \frac{1}{2}(1 + (-1)^n)$$

$$= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

(c) PMF of $X^2$

Let $Y = X^2$ PMF of $Y$ is $P_Y(y) = \begin{cases} 1, & y = 1 \\ 0, & \text{o.w} \end{cases}$

since $x = \pm 1 \rightarrow y = x^2 = 1$